Principal-Centric Reasoning in Constructive Authorization Logic

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Abstract

We present an authorization logic DTL_0 that explicitly relativizes reasoning to beliefs of principals. The logic assumes that principals are conceited in their beliefs. We describe the natural deduction system, sequent calculus, Hilbert-style axiomatization, and Kripke semantics of the logic. We prove several meta-theoretic results including cut-elimination, and soundness and completeness for the Kripke semantics. Translations from several other authorization logics into DTL_0, as well as formal connections between DTL_0 and the modal logic constructive S4 are also presented. Finally, a related logic BL_0 is considered and its properties are studied.

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1 Introduction

Authorization refers to the act of deciding whether or not an agent making a request to perform an operation on a resource should be allowed to do so. For example, the agent may be a browser trying to read pages from a website. In that case, the site's web server may consult the browser's credentials and a `.htaccess` file to determine whether to send the pages or not. Such access control is pervasive in computer systems. As systems and their user environments evolve, policies used for access control may become complex and error prone. This suggests the need for formal mechanisms to represent, enforce, and analyze policies. Logic appears to be a useful mechanism for these purposes. Policies may be expressed as formulas in a suitably chosen logic. This has several merits. First, the logic's rigorous inference eliminates any ambiguity that may be inherent in a textual description of policies. Second, policies may be enforced end-to-end using generic logic-based mechanisms like proof-carrying authorization [8–10, 40]. Third, by writing policies in a logic, there is hope that the policies themselves can be checked for correctness against some given criteria (see e.g., [3, 33, 42, 44]).

Whereas first-order logic and sometimes propositional logic suffice to express many authorization policies, decentralized systems pose a peculiar challenge: how do we express and combine policies of *different* agents and systems? This is often necessary since policies and the authorizations derived from them may vary from system to system. Policies of different users, programs, and systems may also interact to allow or deny access. To model such decentralized policies, Abadi and others proposed logics with formulas of the form $K \mathsf{~says~} A$, where K is an agent or a system (abstractly called a principal) and A is a formula representing a policy [6, 39]. The intended meaning of the formula is that principal K states, or believes that policy A holds. From a logical perspective $K \mathsf{~says~} \cdot$ is a modality and the logic is an indexed modal logic with one modality for each principal. We call such a modal logic an *authorization logic*. In the past fifteen years there have been numerous proposals describing authorization logics that differ widely in the specific axioms (or inference rules) used for $K \mathsf{~says~} \cdot$ [2, 3, 8–10, 21, 23, 25, 31–33, 40, 41]. One emerging trend is the increased use of intuitionistic logics for authorization (e.g., [3, 25, 29, 31–33, 40, 50]) as opposed to classical logics.

This paper presents a new intuitionistic authorization logic called DTL_0. This logic is peculiar in a certain respect: it abandons the usual objectivity in reasoning from hypothesis, relativizing hypothetical reasoning to principals. The hypothetical judgment of the logic has the form $\Gamma \xrightarrow{K} A$, which means, up to a first approximation, that principal K may reason from hypothesis Γ that A is true. While principal K reasons, $K \mathsf{~says~} A$ implies A for each A, thus making all policies local to K available. This may not be true when another principal K' reasons. Reasoning of different principals may interact through the says connective. Although this choice of binding hypothetical reasoning to principals may be unintuitive from a philosophical point of view, it seems quite apt for reasoning about authorization policies.

Our primary interest in developing DTL_0 is deployment in proof-carrying authorization [8–10, 40]. Hence our main focus is DTL_0's proof-theory, especially a natural deduction system (with proof-terms) and a sequent calculus, which we describe in detail. We prove several meta-theoretic properties of both formulations, including cut-elimination for the sequent calculus. We also present a Hilbert-style proof system for DTL_0, and

sound and complete Kripke semantics. The principal-centric reasoning of DTL_0 reflects in the Kripke semantics: worlds are explicitly associated with principals who may view them. This suggests that principals in DTL_0 may be related to nominals from hybrid logic [15, 20, 22]. We also show that DTL_0 is a generalization of constructive modal S4 [7, 46], and describe a sound and complete translation from DTL_0 to multi-modal constructive S4.

Besides investigating the theory of DTL_0, a second goal of this paper is to understand how the logic relates to existing authorization logics, and to the numerous logic-based languages for writing authorization policies (e.g., [11, 24, 37, 47]). In this regard we present simple, syntax directed translations from several families of authorization logics and an authorization language to DTL_0, thus showing that DTL_0 is at least as expressive as each of them. These translations are part of a more ambitious effort to establish a common framework in which policies written in different logics and languages may be combined. Some initial work in this direction using modal S4 as foundation may be found in earlier work [31].

DTL_0 is a fragment of a larger authorization logic, DTL, which we are currently developing. The latter is quite broad, incorporating first-order quantifiers, explicit time for modeling time-bounded policies [25], and linearity for modeling consumable credentials [19, 21, 32]. Detailed investigation of these constructs is the subject of ongoing work. Besides these, there are some other aspects of authorization logics such as compound principals and delegation [6, 31, 39] which we also plan to investigate in the future. On a more practical note, we are implementing a file system with proof-carrying authorization based on DTL. We also plan to develop policy analysis tools using DTL.

By itself, this paper makes three main contributions. First, it presents the logic DTL_0, investigating in detail its proof-theory (Sections 2 and 3). Second, it presents simple, intuitive translations from several existing policy formalisms to DTL_0, thus taking a step towards a common foundation for combining policies represented in different formalisms (Section 5). A third, albeit minor contribution of the paper is sound and complete Kripke semantics (Section 4), which are relatively rare for authorization logics; the only other examples we know of are semantics for lax-like modalities [31], and those for an earlier logic based on the modal logic K [6]. We omit a description of large examples from this paper, leaving them to a separate paper.

2 The logic DTL_0

DTL_0 extends propositional intuitionistic logic with a principal-indexed modality, K says A. Principals, denoted K, are abstractions for users, programs, machines, and systems, that either create policies or request access to resources. We stipulate a fixed set of principals $\mathtt{Prin}$, pre-ordered by a relation written $\succeq$. $K_1 \succeq K_2$ is read "principal K_1 is stronger than principal K_2", and entails that K_1 says A implies K_2 says A for every formula A. We assume that $\mathtt{Prin}$ has at least one maximum element, called the *local authority* (denoted ℓ).[1] The syntax of formulas in DTL_0 is shown below. P denotes

[1] To the best of our understanding, the term *local authority* as used here was first introduced in the preview implementation of the language SecPAL [1].

atomic formulas.

$$A, B, C \ ::= \ P \mid A \wedge B \mid A \vee B \mid \top \mid \bot \mid A \supset B \mid K \ \mathsf{says}\ A$$

Axiomatic Proof-System. A Hilbert-style proof-system for DTL_0 consists of any axiomatization of propositional intuitionistic logic (see Appendix C for one possibility), and the following rules and axioms for K says A. We write $\vdash_H A$ to mean that A is provable without assumptions (i.e., that A is a tautology).

$$\frac{\vdash_H A}{\vdash_H K \ \mathsf{says}\ A} \qquad \text{(nec)}$$

$$\begin{array}{ll}
\vdash_H (K \ \mathsf{says}\ (A \supset B)) \supset ((K \ \mathsf{says}\ A) \supset (K \ \mathsf{says}\ B)) & \text{(K)} \\
\vdash_H (K \ \mathsf{says}\ A) \supset K \ \mathsf{says}\ K \ \mathsf{says}\ A & \text{(4)} \\
\vdash_H K \ \mathsf{says}\ ((K \ \mathsf{says}\ A) \supset A) & \text{(C)} \\
\vdash_H (K_1 \ \mathsf{says}\ A) \supset (K_2 \ \mathsf{says}\ A) \text{ if } K_1 \succeq K_2. & \text{(S)}
\end{array}$$

(nec) and (K) are the usual necessitation rule and closure under consequence axiom for normal modal logics (see e.g., [14]). (4) is also standard from modal logics such as S4. (C) is the characterizing axiom of DTL_0. It has been used to characterize *conceited* reasoners in doxastic logic (hence the name C) [48]. Intuitively, the axiom means that every principal says that all its statements are true. Although the propriety of this axiom in the context of doxastic reasoning has been questioned (e.g., [48]), it seems quite useful for authorization. The axiom (S) means that whenever principal K_1 believes a formula A, every weaker principal K_2 believes it as well.

The following properties may be established in DTL_0. $\not\vdash_H A$ means that A is not valid in the stated generality (although specific instances of A may be valid). $A \equiv B$ denotes $(A \supset B) \wedge (B \supset A)$.

$$\vdash_H (\ell \ \mathsf{says}\ A) \supset (K \ \mathsf{says}\ A)$$

$$\vdash_H (K \ \mathsf{says}\ K \ \mathsf{says}\ A) \equiv (K \ \mathsf{says}\ A)$$

$$\not\vdash_H A \supset K \ \mathsf{says}\ A$$

$$\not\vdash_H (K \ \mathsf{says}\ A) \supset A$$

$$\vdash_H (K \ \mathsf{says}\ (A \wedge B)) \equiv ((K \ \mathsf{says}\ A) \wedge (K \ \mathsf{says}\ B))$$

$$\not\vdash_H (K \ \mathsf{says}\ (A \vee B)) \supset ((K \ \mathsf{says}\ A) \vee (K \ \mathsf{says}\ B))$$

$$\not\vdash_H \bot$$

$$\not\vdash_H (K \ \mathsf{says}\ A) \supset (K' \ \mathsf{says}\ K \ \mathsf{says}\ A)$$

Defined Connectives. The last property above means that if K says A, not every principal K' may believe this. In some cases, this may not be desirable, since some policies may be stated and *published* by K and in these cases we may expect that K' says K says A. In particular, if K issues a credential containing a policy, we may want that the policy be believed by all principals. Further, there may some policies

that are believed by all principals. To model such published and shared policies, we introduce two defined connectives in the logic. The first connective, $\mathsf{global}\ A$, implies $K\ \mathsf{says}\ A$ for each principal K, and may be understood as the statement that A is a common belief of all principals. The second connective, $K\ \mathsf{publ}\ A$ (read K publishes A) implies $K'\ \mathsf{says}\ K\ \mathsf{says}\ A$ for each K', and intuitively means that K *publishes the fact that it believes* A. We define,

$$\begin{aligned} \mathsf{global}\ A &\stackrel{\text{def}}{=} \ell\ \mathsf{says}\ A \\ K\ \mathsf{publ}\ A &\stackrel{\text{def}}{=} \mathsf{global}\ (K\ \mathsf{says}\ A) \end{aligned}$$

It is easy to check that the following hold.

$\vdash_H (\mathsf{global}\ A) \supset K\ \mathsf{says}\ A$

$\vdash_H (\mathsf{global}\ A) \supset K\ \mathsf{publ}\ A$

$\vdash_H (K\ \mathsf{publ}\ A) \supset K\ \mathsf{says}\ A$

$\vdash_H (K\ \mathsf{publ}\ A) \supset K'\ \mathsf{says}\ (K\ \mathsf{says}\ A)$

$\not\vdash_H (K\ \mathsf{says}\ A) \supset K\ \mathsf{publ}\ A$

Example 2.1 (Policies in DTL_0)**.** We illustrate the use of DTL_0 for expressing authorization policies through a simple example. Suppose that the principal OAL (Online Academic Library) represents an online repository of scientific articles. Academics institutions (such as CMU) may buy corporate subscriptions that allow all their members to download articles from OAL. It is up to the subscribing institutions to tell OAL who their members are. Alice is an individual who wishes to download an article from OAL. Let the formula `downloadAlice` mean that Alice may download articles from OAL, and let `memberAliceCMU` mean that Alice is a member of CMU. Further, let us assume that CMU has a subscription at OAL. The following represent possible policies of the principals.

1. OAL says ((CMU says `memberAliceCMU`) $\supset$ `memberAliceCMU`)
2. OAL says (`memberAliceCMU` $\supset$ `downloadAlice`)
3. CMU publ `memberAliceCMU`

The first policy, stated by OAL, means that if CMU says that Alice is its member, then this is the case. The second policy, also stated by OAL, means that if Alice is a member of CMU, then she may download articles. The third policy, stated and published by CMU, means that Alice is a member of CMU. It is easy to check that these three policies entail the formula OAL says `downloadAlice` in DTL_0, and that this would not be the case if we changed publ to says in the last policy.

3 Structural Proof Theory

Next we develop the structural proof theory of DTL_0, namely a natural deduction system and a sequent calculus. Besides explaining the meanings of connectives precisely, the natural deduction formulation provides a syntax for proof terms that are a basis for proof-carrying authorization (our intended deployment for DTL_0). The sequent calculus is necessary to prove some of the theorems in later sections. We also expect that the sequent calculus will be useful in proof-construction, which is also essential for proof-carrying authorization.

We follow Martin-Löf's judgmental method in developing the structural proof theory, and maintain a strong distinction between formulas and judgments [43]. The presentation of the natural deduction system is more directly based on Pfenning and Davies' work on constructive S4 [46], whereas the presentation of the sequent calculus is inspired by previous work of the author and others on multi-modal S4, also done in the context of access control [32]. In Section 3.6 we show that the natural deduction system, the sequent calculus, and the axiomatic system described earlier are equivalent.

3.1 Natural Deduction

In Martin-Löf's approach to type-theory and logic, formulas are distinguished from judgments. The latter are the objects of knowledge that may be established through proofs. Formulas are the subjects of judgments. For DTL_0, we use two basic (categorical) judgments: A true, meaning that formula A is true, and K claims A, meaning that principal K believes or claims that formula A is true. The two categorical judgments do not entail each other in general. We often abbreviate A true to A, if it is clear from context that we mean the judgment A true and not the formula A.

Of course, in order to represent policies, it is necessary to combine claims of principals using connectives. Since judgments are distinct from formulas, and connectives only apply to the latter, we cannot use the judgment K claims A directly for in such representations. Accordingly, we *internalize* the judgment K claims A into the syntax of formulas as K says A. In other words the judgments (K says A) true and K claims A are equivalent. Since K says A is a formula, it may be combined with other connectives.

Hypothetical Judgments

Reasoning from hypothesis or assumptions is a basic tenet of logic. Logics invariably allow *hypothetical judgments* of the form $\Gamma \vdash A$, meaning that the assumptions in Γ entail formula A. A distinguishing characteristic of DTL_0 is that hypothetical reasoning is always performed relative to the beliefs of a principal K, which we indicate in the hypothetical judgment; we write $\Gamma \vdash^K A$.[2] Formally, K is called the *context* of the hypothetical judgment, or the context of reasoning. The hypothesis are a (possibly empty) multiset of categorical judgments:

$$\Gamma \ ::= \ \cdot \ \mid \Gamma, C \ \mathsf{true} \ \mid \Gamma, K' \ \mathsf{claims}\ C$$

Reasoning in DTL_0 is guided by three basic principles. The first principle, called the *context principle*, describes how the context K affects reasoning.

[2] A represents the judgment A true, not the formula A, but we usually elide the judgment name true.

Context principle. While reasoning in context K, the assumption K' claims A entails A true if $K' \succeq K$.

We incorporate this principle into the natural deduction system by the following rule of inference.

$$\frac{K' \succeq K}{\Gamma, K' \ \mathsf{claims}\ A \vdash^{K} A}\text{claims}$$

Based on the context principle, we may define the meaning of the hypothetical judgment $\Gamma \vdash^{K} A$ precisely as follows:

> *"Assuming that beliefs of principals stronger than K are true, the hypothesis Γ logically entail that A is true".*

Although this choice of relativizing hypothetical judgments to beliefs of principals is non-standard, it seems quite useful from the perspective of access control, where an authorization may succeed or fail, depending on the policies applicable in the surrounding context.

Our second guiding principle, called the *substitution principle*, elaborates the meaning of hypothesis. It states that a hypothesis A true used in a proof may be substituted by an actual proof of the hypothesis.

Substitution principle. $\Gamma \vdash^{K} A$ and $\Gamma, A \vdash^{K} C$ imply $\Gamma \vdash^{K} C$

Unlike the context principle which is incorporated directly as a rule in the natural deduction system, the substitution principle is established as a theorem.

Our third guiding principle, called the *claim principle*, defines the relation between the judgments K claims A and A true. Informally it states that K claims A holds if we can establish A true in context K from the claims of principals stronger than K. Formally, we define an operator $\Gamma|_K$ that restricts the hypothesis Γ to the claims of principals stronger than K.

$$\Gamma|_K \;=\; \{(K' \ \mathsf{claims}\ C) \in \Gamma \mid K' \succeq K\}$$

The claim principle may then be written as follows.

Claim principle. $\Gamma|_K \vdash^{K} A$ and Γ, K claims $A \vdash^{K'} C$ imply $\Gamma \vdash^{K'} C$.

Like the substitution principle, the claim principle is admissible as a theorem in the natural deduction system. In fact, we prove the two principles simultaneously in a single theorem (Theorem 3.2).

Inference Rules

The inference rules of the natural deduction system are summarized in Figure 1. The most basic inference rule is (hyp). It means that if A true is a hypothesis, then A must be true.

$$\frac{}{\Gamma, A \vdash^{K} A}\text{hyp}$$

The rule (claims) captures the context principle as described earlier. The remaining rules are directed by the connectives of DTL_0. For each connective, there are *introduction*

$$\frac{}{\Gamma, A \vdash^K A}\text{hyp} \qquad \frac{K' \succeq K}{\Gamma, K' \;\mathsf{claims}\; A \vdash^K A}\text{claims}$$

$$\frac{\Gamma|_K \vdash^K A}{\Gamma \vdash^{K'} K \;\mathsf{says}\; A}\mathsf{says}\text{I} \qquad \frac{\Gamma \vdash^{K'} K \;\mathsf{says}\; A \qquad \Gamma, K \;\mathsf{claims}\; A \vdash^{K'} C}{\Gamma \vdash^{K'} C}\mathsf{says}\text{E}$$

$$\frac{\Gamma \vdash^K A \qquad \Gamma \vdash^K B}{\Gamma \vdash^K A \wedge B}\wedge\text{I} \qquad \frac{\Gamma \vdash^K A \wedge B}{\Gamma \vdash^K A}\wedge\text{E}_1 \qquad \frac{\Gamma \vdash^K A \wedge B}{\Gamma \vdash^K B}\wedge\text{E}_2$$

$$\frac{\Gamma \vdash^K A}{\Gamma \vdash^K A \vee B}\vee\text{I}_1 \qquad \frac{\Gamma \vdash^K B}{\Gamma \vdash^K A \vee B}\vee\text{I}_2 \qquad \frac{\Gamma \vdash^K A \vee B \qquad \Gamma, A \vdash^K C \qquad \Gamma, B \vdash^K C}{\Gamma \vdash^K C}\vee\text{E}$$

$$\frac{}{\Gamma \vdash^K \top}\top\text{I} \qquad \frac{\Gamma \vdash^K \bot}{\Gamma \vdash^K C}\bot\text{E} \qquad \frac{\Gamma, A \vdash^K B}{\Gamma \vdash^K A \supset B}\supset\text{I} \qquad \frac{\Gamma \vdash^K A \supset B \qquad \Gamma \vdash^K A}{\Gamma \vdash^K B}\supset\text{E}$$

Figure 1: Natural Deduction for DTL_0

rules (marked I) that specify how a proof of the connective may be constructed directly, and *elimination* rules (marked E) that specify how a proof of the connective may be used. In the following we describe briefly the rules for says.

How can we establish (K says A) true? Since (K says A) true is equivalent to K claims A, the claim principle tells us that (K says A) true may be established if we can establish A true in context K using assumptions of principals stronger than K. This is exactly what the rule (saysI) captures:

$$\frac{\Gamma|_K \vdash^K A}{\Gamma \vdash^{K'} K \;\mathsf{says}\; A}\mathsf{says}\text{I}$$

Dually, how can we use the fact (K says A) true? Again, since (K says A) true and K claims A are equivalent, from the fact (K says A) true, we should be able to assume K claims A. This is captured by the elimination rule (saysE):

$$\frac{\Gamma \vdash^{K'} K \;\mathsf{says}\; A \qquad \Gamma, K \;\mathsf{claims}\; A \vdash^{K'} C}{\Gamma, \vdash^{K'} C}\mathsf{says}\text{E}$$

Rules for the connectives $\wedge$, $\vee$, $\top$, $\bot$, and $\supset$ are standard, with the exception that there is a context associated with each hypothetical judgment. We elide a description of these standard rules.

3.2 Meta-Theory of the Natural Deduction System

Having seen all the rules of the natural deduction system, we now seek to prove that the substitution and context principles are admissible in DTL_0. Before doing that, we establish another fundamental property called *subsumption* that is needed to complete the proof. Subsumption states that weaker contexts make more formulas provable. Intuitively, this follows from the definition of hypothetical judgments.

Theorem 3.1 (Subsumption). *$K \succeq K'$ and $\Gamma \vdash^K A$ imply $\Gamma \vdash^{K'} A$.*

Proof. By induction on the derivation of $\Gamma \vdash^K A$. $\square$

The following theorem formally states that both the substitution and claim principles hold.

Theorem 3.2 (Substitution and Claim). *The following hold.*

1. $\Gamma \vdash^K A$ *and* $\Gamma, A\ \mathsf{true} \vdash^K C$ *imply* $\Gamma \vdash^K C$.
2. $\Gamma|_K \vdash^K A$ *and* $\Gamma, K\ \mathsf{claims}\ A \vdash^{K'} C$ *imply* $\Gamma \vdash^{K'} C$.

Proof. By simultaneous induction on the second given derivations. In the case of (2), rule (claims) we use Theorem 3.1. $\square$

3.3 Proof Terms

The natural deduction system described above may be augmented with proof terms in the usual way. We use standard notation from the λ calculus for denoting most parts of proof terms; new notation is introduced only for the introduction and elimination forms of says. The syntax of proof terms is summarized below. x, y denote variables.

$$\begin{array}{rcl} t & ::= & x \mid \lambda x.t \mid t_1\ t_2 \mid \langle t_1, t_2\rangle \mid \mathtt{proj}_1\, t \mid \mathtt{proj}_2\, t \mid \langle\rangle \mid \mathtt{abort}\, t \\ & & \mathtt{inl}\, t \mid \mathtt{inr}\, t \mid \mathtt{case}(t, x.t_1, y.t_2) \mid \{t\}_K \mid t_1 \Rightarrow x.t_2 \end{array}$$

The constructors $\{t\}_K$ and $t_1 \Rightarrow x.t_2$ are the introduction and elimination forms for $K\ \mathsf{says}\ A$. The variables x, y in $\lambda x.t$, $\mathtt{case}(t, x.t_1, y.t_2)$, and $t_1 \Rightarrow x.t_2$ are bound. We identify terms up to α-renaming of such bound variables.

Figure 2 shows the modified inference rules with proof terms. As usual, we name all assumptions in Γ by associating unique variables with them. We do not need to syntactically distinguish between variables associated with assumptions $A\ \mathsf{true}$ and those associated with assumptions $K\ \mathsf{claims}\ A$. Hypothetical judgments are augmented with proof terms; they take the form $\Gamma \vdash^K t : A$. The definition of $\Gamma|_K$ is lifted to include variables: $\Gamma|_K = \{(x : K'\ \mathsf{claims}\ C) \in \Gamma \mid K' \succeq K\}$.

Once again, we can prove a subsumption principle:

Theorem 3.3 (Subsumption). *$K \succeq K'$ and $\Gamma \vdash^K t : A$ imply $\Gamma \vdash^{K'} t : A$.*

Proof. By induction on the derivation of $\Gamma \vdash^K t : A$. $\square$

Let $[t_1/x]t_2$ denote the capture avoiding substitution of t_1 for all occurrences of x in t_2. (We elide the obvious definition.) The substitution and claim principles (Theorem 3.2) may be modified, obtaining the following new principles.

Theorem 3.4 (Substitution and Claim). *The following hold.*

1. $\Gamma \vdash^K t_1 : A$ *and* $\Gamma, x : A\ \mathsf{true} \vdash^K t_2 : C$ *imply* $\Gamma \vdash^K [t_1/x]t_2 : C$.
2. $\Gamma|_K \vdash^K t_1 : A$ *and* $\Gamma, x : K\ \mathsf{claims}\ A \vdash^{K'} t_2 : C$ *imply* $\Gamma \vdash^{K'} [t_1/x]t_2 : C$.

Proof. By simultaneous induction on the second given derivations. $\square$

$$\frac{}{\Gamma, x : A \vdash^K x : A}\text{hyp} \qquad \frac{K' \succeq K}{\Gamma, x : K' \textsf{ claims } A \vdash^K x : A}\text{claims}$$

$$\frac{\Gamma|_K \vdash^K t : A}{\Gamma \vdash^{K'} \{t\}_K : K \textsf{ says } A}\textsf{saysI} \qquad \frac{\Gamma \vdash^{K'} t_1 : K \textsf{ says } A \qquad \Gamma, x : K \textsf{ claims } A \vdash^{K'} t_2 : C}{\Gamma \vdash^{K'} t_1 \Rightarrow x.t_2 : C}\textsf{saysE}$$

$$\frac{\Gamma \vdash^K t_1 : A \qquad \Gamma \vdash^K t_2 : B}{\Gamma \vdash^K \langle t_1, t_2 \rangle : A \wedge B}\wedge\text{I} \qquad \frac{\Gamma \vdash^K t : A \wedge B}{\Gamma \vdash^K \texttt{proj}_1\, t : A}\wedge\text{E}_1 \qquad \frac{\Gamma \vdash^K t : A \wedge B}{\Gamma \vdash^K \texttt{proj}_2\, t : B}\wedge\text{E}_2$$

$$\frac{\Gamma \vdash^K t : A}{\Gamma \vdash^K \texttt{inl}\, t : A \vee B}\vee\text{I}_1 \qquad \frac{\Gamma \vdash^K t : B}{\Gamma \vdash^K \texttt{inr}\, t : A \vee B}\vee\text{I}_2$$

$$\frac{\Gamma \vdash^K t : A \vee B \qquad \Gamma, x : A \vdash^K t_1 : C \qquad \Gamma, y : B \vdash^K t_2 : C}{\Gamma \vdash^K \texttt{case}(t, x.t_1, y.t_2) : C}\vee\text{E}$$

$$\frac{}{\Gamma \vdash^K \langle\rangle : \top}\top\text{I} \qquad \frac{\Gamma \vdash^K t : \bot}{\Gamma \vdash^K \texttt{abort}\, t : C}\bot\text{E}$$

$$\frac{\Gamma, x : A \vdash^K t : B}{\Gamma \vdash^K \lambda x.t : A \supset B}\supset\text{I} \qquad \frac{\Gamma \vdash^K t_1 : A \supset B \qquad \Gamma \vdash^K t_2 : A}{\Gamma \vdash^K t_1\ t_2 : B}\supset\text{E}$$

Figure 2: Proof terms for DTL_0

Local Reduction and Local Expansion

Pfenning and Davies proposed two general principles to verify that the inference rules in a natural deduction system fit well with each other [46]. They called these principles local soundness and local completeness. In the following we present these principles for DTL_0 with proof terms. Analogous principles may also be obtained at the level of proofs instead of proof terms.

Briefly, local soundness states that if the introduction of a connective is immediately followed by its elimination in a proof, then it should be possible to *locally reduce* the proof by eliminating this detour. Local soundness for a connective guarantees that the elimination rule(s) for the connective are not too strong, i.e., they do not conclude any formula that would not already follow from the inputs to the introduction rule(s) of the connective. The dual principle, local completeness, states that given any proof of a formula, it should be possible to *locally expand* the proof by eliminating its top level connective and re-introducing it, obtaining a bigger proof of the original formula. Local completeness for a connective guarantees that its elimination rule(s) are strong enough to conclude everything that is needed to re-constitute a proof of the connective from its introduction rule(s). Together the two principles provide assurance that the introduction and elimination rules are in harmony with each other.

Under the Curry-Howard isomorphism, local reduction and local expansion correspond to the familiar concepts of β-reduction and η-expansion, respectively. We present type-directed variants of β-reduction and η-expansion for DTL_0 in Figure 3. In addition

$\boxed{\beta\text{-reduction}}$

$$\frac{\Gamma \vdash^K t_1 : A \qquad \Gamma \vdash^K t_2 : B}{\Gamma \vdash^K \texttt{proj}_1\ \langle t_1, t_2\rangle \rightsquigarrow_\beta t_1 : A} \qquad \frac{\Gamma \vdash^K t_1 : A \qquad \Gamma \vdash^K t_2 : B}{\Gamma \vdash^K \texttt{proj}_2\ \langle t_1, t_2\rangle \rightsquigarrow_\beta t_2 : B}$$

$$\frac{\Gamma \vdash^K t : A \qquad \Gamma, x : A \vdash^K t_1 : C \qquad \Gamma, y : B \vdash^K t_2 : C}{\Gamma \vdash^K \texttt{case}(\texttt{inl}\ t, x.t_1, y.t_2) \rightsquigarrow_\beta [t/x]t_1 : C}$$

$$\frac{\Gamma \vdash^K t : B \qquad \Gamma, x : A \vdash^K t_1 : C \qquad \Gamma, y : B \vdash^K t_2 : C}{\Gamma \vdash^K \texttt{case}(\texttt{inr}\ t, x.t_1, y.t_2) \rightsquigarrow_\beta [t/y]t_2 : C}$$

$$\frac{\Gamma, x : A \vdash^K t_1 : B \qquad \Gamma \vdash^K t_2 : A}{\Gamma \vdash^K (\lambda x.t_1)\ t_2 \rightsquigarrow_\beta [t_2/x]t_1 : B} \qquad \frac{\Gamma|_K \vdash^K t_1 : A \qquad \Gamma, x : K \textsf{ claims } A \vdash^{K'} t_2 : C}{\Gamma \vdash^{K'} (\{t_1\}_K \Rightarrow x.t_2) \rightsquigarrow_\beta [t_1/x]t_2 : C}$$

$\boxed{\eta\text{-expansion}}$

$$\frac{\Gamma \vdash^K t : A \wedge B}{\Gamma \vdash^K t \rightsquigarrow_\eta \langle \texttt{proj}_1\ t, \texttt{proj}_2\ t\rangle : A \wedge B} \qquad \frac{\Gamma \vdash^K t : A \vee B}{\Gamma \vdash^K t \rightsquigarrow_\eta \texttt{case}(t, x.\ \texttt{inl}\ x, y.\ \texttt{inr}\ y) : A \vee B}$$

$$\frac{\Gamma \vdash^K t : \top}{\Gamma \vdash^K t \rightsquigarrow_\eta \langle\rangle : \top} \qquad \frac{\Gamma \vdash^K t : \bot}{\Gamma \vdash^K t \rightsquigarrow_\eta \texttt{abort}\ t}$$

$$\frac{\Gamma \vdash^K t : A \supset B}{\Gamma \vdash^K t \rightsquigarrow_\eta \lambda x.(t\ x) : A \supset B}(x \notin \Gamma) \qquad \frac{\Gamma \vdash^{K'} t : K \textsf{ says } A}{\Gamma \vdash^{K'} t \rightsquigarrow_\eta (t \Rightarrow x.\{x\}_K) : K \textsf{ says } A}$$

Figure 3: Basic rules for β-reduction and η-expansion

to these rules, there are a number of congruence rules, which we list in Appendix A. Our β-reduction and η-expansion rules are somewhat unusual since they include type information, and apply only to well-typed terms. For example, β-reduction has the form $\Gamma \vdash^K t \rightsquigarrow_\beta t' : A$, meaning that the proof term t (proving A $\textsf{true}$ under hypothesis Γ in context K) β-reduces to t'. We prove separately (see Theorem 3.5 below) that if $\Gamma \vdash^K t \rightsquigarrow_\beta t' : A$, then $\Gamma \vdash^K t : A$ and $\Gamma \vdash^K t' : A$. This theorem subsumes the usual type-preservation or subject reduction theorem. The treatment of η-expansion is similar. It is easy to see that on well-typed terms without the constructors $\{t\}_K$ and $t_1 \Rightarrow x.t_2$ our definitions of β-reduction and η-expansion coincide with conventional (untyped) definitions.

Theorem 3.5 (Typing)**.** *For* $\rightsquigarrow \in \{\rightsquigarrow_\beta, \rightsquigarrow_\eta\}$, *if* $\Gamma \vdash^K t \rightsquigarrow t' : A$ *then,*

1. $\Gamma \vdash^K t : A$
2. $\Gamma \vdash^K t' : A$

Proof. In each case by induction on the given derivation of $\Gamma \vdash^K t \rightsquigarrow t' : A$. □

$$\frac{P \text{ atomic}}{\Gamma, P \xrightarrow{K} P}\text{init} \qquad \frac{\Gamma, K\ \mathsf{claims}\ A, A \xrightarrow{K'} C \qquad K \succeq K'}{\Gamma, K\ \mathsf{claims}\ A \xrightarrow{K'} C}\text{claims}$$

$$\frac{\Gamma|_K \xrightarrow{K} A}{\Gamma \xrightarrow{K'} K\ \mathsf{says}\ A}\mathsf{says}\text{R} \qquad \frac{\Gamma, K\ \mathsf{says}\ A, K\ \mathsf{claims}\ A \xrightarrow{K'} C}{\Gamma, K\ \mathsf{says}\ A \xrightarrow{K'} C}\mathsf{says}\text{L}$$

$$\frac{\Gamma \xrightarrow{K} A \qquad \Gamma \xrightarrow{K} B}{\Gamma \xrightarrow{K} A \wedge B}\wedge\text{R} \qquad \frac{\Gamma, A \wedge B, A, B \xrightarrow{K} C}{\Gamma, A \wedge B \xrightarrow{K} C}\wedge\text{L}$$

$$\frac{\Gamma \xrightarrow{K} A}{\Gamma \xrightarrow{K} A \vee B}\vee \text{R}_1 \qquad \frac{\Gamma \xrightarrow{K} B}{\Gamma \xrightarrow{K} A \vee B}\vee \text{R}_2 \qquad \frac{\Gamma, A \vee B, A \xrightarrow{K} C \qquad \Gamma, A \vee B, B \xrightarrow{K} C}{\Gamma, A \vee B \xrightarrow{K} C}\vee \text{L}$$

$$\frac{}{\Gamma \xrightarrow{K} \top}\top\text{R} \qquad \frac{}{\Gamma, \bot \xrightarrow{K} C}\bot\text{L}$$

$$\frac{\Gamma, A \xrightarrow{K} B}{\Gamma \xrightarrow{K} A \supset B}\supset\text{R} \qquad \frac{\Gamma, A \supset B \xrightarrow{K} A \qquad \Gamma, A \supset B, B \xrightarrow{K} C}{\Gamma, A \supset B \xrightarrow{K} C}\supset\text{L}$$

Figure 4: Sequent calculus for DTL_0

3.4 Sequent Calculus

Next, we describe a sequent calculus for DTL_0. As in the natural deduction system, we maintain a distinction between formulas and judgments. The categorical and hypothetical judgments used in the sequent calculus are the same as those in the natural deduction system. To avoid confusion with the natural deduction system, we write hypothetical judgments in the sequent calculus as $\Gamma \xrightarrow{K} A$, and call them *sequents*.[3] The inference rules of the sequent calculus are shown in Figure 4. With the exception of (init) and (claims), all rules are directed by the connectives of DTL_0. For each connective we have right rules which describe how the connective may be inferred as the conclusion of the sequent, and left rules which specify how the connective may be used as a hypothesis.

Rule (init) states that if we assume that an atomic formula P is true, then in any context K we may conclude that P is true. For non-atomic formulas, we prove a corresponding result as a theorem (see Theorem 3.8). The rules (claims), (saysR), and (saysL) characterize DTL_0. Read from the conclusion to the premises, rule (claims) states that whenever we assume $K\ \mathsf{claims}\ A$, we are also justified in assuming that A is true, if we are reasoning in a context K' such that $K \succeq K'$. This captures the context principle described earlier.

Rule (saysR) is analogous to the rule (saysI) from the natural deduction system and means that $K\ \mathsf{says}\ A$ may be established in any context if we can prove in context K

[3]Technically, in the sequent calculus there is a distinction between *hypothesis* $A\ \mathsf{true}$ and *conclusions* $A\ \mathsf{true}$; they are distinct categorical judgments. However, this distinction is always evident from the positions of the judgments in sequents, and we avoid separating the two in syntax.

that A is true using only the claims of principals stronger than K. Rule (saysL) captures the idea that K says A internalizes K claims A: if we assume that K says A is true, then we may also assume K claims A. The rules for the connectives $\wedge$, $\vee$, $\top$, $\bot$, and $\supset$ are standard, except for a context which is associated with each sequent.

3.5 Meta-Theory of the Sequent Calculus

The sequent calculus described above enjoys several meta-theoretic properties. For example, it is evident from the rules in Figure 4 that the sequent calculus enjoys the *subformula property*, i.e., any formula occurring in the proof of a sequent must occur inside the formulas of the sequent. Several structural properties such as *weakening* also hold in the sequent calculus. As for the natural deduction system, we may also establish a subsumption principle for the sequent calculus.

Theorem 3.6 (Subsumption). $\Gamma \xrightarrow{K} A$ *and* $K \succeq K'$ *imply* $\Gamma \xrightarrow{K'} A$.

Proof. By induction on the given derivation of $\Gamma \xrightarrow{K} A$. See Appendix B for details. □

A very important property of the sequent calculus is cut-elimination [35]. This property is analogous to the substitution principle and the claim principle; formally it states that adding a cut rule to a sequent calculus does not make more judgments provable. More generally, it implies that all (natural deduction) proofs can be normalized. Besides providing assurance of the logic's strong foundation, proof normalization is sometimes useful for auditing proofs. Instead of stating explicitly the rules of cut for our sequent calculus and showing that they may be eliminated, we prove the following theorem which states that cut principles analogous to the substitution principle and the claim principle are admissible in the sequent calculus.

Theorem 3.7 (Admissibility of Cut). *The following cut principles hold for the sequent calculus of Figure 4.*

1. $\Gamma \xrightarrow{K} A$ *and* $\Gamma, A \xrightarrow{K} C$ *imply that* $\Gamma \xrightarrow{K} C$.
2. $\Gamma|_K \xrightarrow{K} A$ *and* $\Gamma, K\ \mathsf{claims}\ A \xrightarrow{K'} C$ *imply that* $\Gamma \xrightarrow{K'} C$.

Proof. Both statements can be proved simultaneously by lexicographic induction, first on the size of the cut judgments (A true or K claims A), and then on the size of the two given derivations, as in earlier work [45]. See Appendix B for details. □

The logical dual of the cut-elimination theorem is the following identity theorem, which states that whenever A true is assumed as a hypothesis, we may conclude it. This generalizes the (init) rule from atomic to arbitrary formulas.

Theorem 3.8 (Identity). *For each formula* A, $\Gamma, A \xrightarrow{K} A$.

Proof. By induction on A. See Appendix B for details. □

3.6 Equivalence

An obvious question is whether the axiomatic system, natural deduction system, and sequent calculus presented for DTL_0 validate the same judgments. The following theorem shows that the natural deduction system and sequent calculus validate exactly the same judgments, and that they can be embedded trivially into the axiomatic system.

Theorem 3.9 (Equivalence)**.** *The following are equivalent for any K and A.*

1. $\cdot \vdash^K A$ *in the natural deduction system.*

2. $\cdot \xrightarrow{K} A$ *in the sequent calculus.*

3. $\vdash_H K\ \mathsf{says}\ A$ *in the axiomatic system.*

Proof. See Appendix C. $\square$

Observe that there is no equivalent of $\vdash_H B$ in the sequent calculus (or natural deduction system) unless B has the form $K\ \mathsf{says}\ A$. In this sense, the above theorem actually *embeds* the sequent calculus into the axiomatic system. While it is possible to recover the entire axiomatic system in the sequent calculus by adding non-indexed hypothetical judgments $\Gamma \longrightarrow A$, this extension seems uninteresting for authorization policies, and we omit it.

4 Kripke Semantics for DTL_0

Next we describe sound and complete Kripke semantics for DTL_0. Although not directly applicable to policies, Kripke semantics are an invaluable tool for proving properties of the logic (e.g., [4, 31]). There is also hope that Kripke countermodels can be used as proofs of *failure*, in case an authorization does not succeed. Our presentation of Kripke semantics is inspired by work on the modal logic constructive S4 [7], and also uses some ideas from work on Kripke semantics of lax logic [27, 31].

The distinguishing characteristic of our Kripke semantics are *views* [31]. With each world w, we associate a set of principals $\theta(w)$ to whom the world is said to be visible. Our correctness property is that $\cdot \xrightarrow{K} A$ if and only if *each world visible to K satisfies A*.[4] In this manner, views allow us to distinguish reasoning in one context from that in another. If $K \succeq K'$ then we require that any world visible to K' also be visible to K. This ensures that context K validates fewer formulas than context K', and captures the subsumption principle (Theorem 3.6).

We model falsehood by explicitly specifying in each frame a set F of worlds where $\bot$ holds. These worlds are called fallible worlds [26, 27, 49]. We say that $w \models \bot$ iff $w \in F$. To model intuitionistic implication, we use a pre-order $\leq$ between worlds (as usual) and say that $w \models A \supset B$ iff for all w', $w \leq w'$ and $w' \models A$ imply $w' \models B$. Finally, to model the modality says, we use a principal-indexed binary relation $\sqsubseteq_K$ between worlds and define:

[4] Throughout this section and the next, we use the sequent calculus of DTL_0 to state correctness properties. Use of the sequent calculus as opposed to the natural deduction system or the axiomatic system is partly a matter of personal taste and partly a matter of technical convenience.

$$w \models K \ \mathsf{says}\ A \text{ iff either } w \in F \text{ or for all } w', w'', w \leq w' \sqsubseteq_K w'' \text{ implies } w'' \models A.$$

The clause $w \in F$ in the above definition is required to validate $\bot \supset K$ says A. The remaining definition is a generalization of satisfaction for $\Box A$ from Kripke semantics of constructive S4 [7]. To validate axiom (4), we stipulate that $\sqsubseteq_K; \leq$ be a subset of $\sqsubseteq_K$.

Both the use of a pre-order to model intuitionistic implication, and the use of different binary relations to model each modality are standard in modal logic. The novelty here is the interaction of these relations with views. We require that $\leq$ preserve views, i.e., if $w \leq w'$ and w be visible to K, then w' also be visible to K. We also require that whenever $w \sqsubseteq_K w'$, w' be visible to K. For example, in the definition of $w \models K$ says A above, w'' would be visible to K. By forcing these restrictions, we ensure that the semantics of all connectives except K says $\cdot$ can be defined without changing views. On the other hand, the semantics of K says $\cdot$ shift the reasoning to worlds that are visible to K. This subtle interaction between views and binary relations captures the exact meaning of formulas in DTL_0.

Definition 4.1 (Kripke Models)**.** A Kripke model M for DTL_0 is a tuple $(W, \theta, \leq, (\sqsubseteq_K)^{K \in \mathtt{Prin}}, \rho, F)$, where

- W is a non-empty set of worlds (worlds are denoted w).
- $\theta : W \mapsto 2^{\mathtt{Prin}}$ is a *view function* that maps each world w to a set of principals. If $K \in \theta(w)$, we say that w is visible to K, else w is said to be invisible to K. We often write W^K for the set $\{w \in W \mid K \in \theta(w)\}$. We require that:

  (View-closure) $K \in \theta(w)$ and $K' \succeq K$ imply $K' \in \theta(w)$.
- $\leq$ is a pre-order on W called the *implication relation*. We require that:

  (Imp-mon) $w \leq w'$ imply $\theta(w) \subseteq \theta(w')$.
- For each K, $\sqsubseteq_K$ is a subset of $W \times W^K$ called the *modality relation*. We require that:

  (Mod-refl) If $w \in W^K$, then $w \sqsubseteq_K w$.

  (Mod-trans) $\sqsubseteq_K$ be transitive.

  (Mod-closure) $w \sqsubseteq_K w'$ and $K' \succeq K$ imply $w \sqsubseteq_{K'} w'$

  (Commutativity) If $w \sqsubseteq_K w' \leq w''$, then $w \sqsubseteq_K w''$.
- $\rho : W \mapsto 2^{\mathtt{AtomicFormulas}}$ is a *valuation function* that maps each world to the set of atomic formulas that hold in it. We require that:

  (Rho-her) $P \in \rho(w)$ and $w \leq w'$ imply $P \in \rho(w')$.
- $F \subseteq W$ is the set of *fallible worlds*. We require that:

  (F-her) $w \in F$ and $w \leq w'$ imply $w' \in F$.

  (F-univ) $w \in F$ imply $P \in \rho(w)$

Definition 4.2 (Satisfaction)**.** Given a model $M = (W, \theta, \leq, (\sqsubseteq_K)^{K \in \texttt{Prin}}, \rho, F)$, and a world $w \in W$, the satisfaction relation $w \models A$ (world w satisfies formula A) is defined by induction on A as follows.

$w \models P$ iff $P \in \rho(w)$.

$w \models A \wedge B$ iff $w \models A$ and $w \models B$.

$w \models A \vee B$ iff $w \models A$ or $w \models B$.

$w \models \top$.

$w \models \bot$ iff $w \in F$.

$w \models A \supset B$ iff for all w', $w \leq w'$ and $w' \models A$ imply $w' \models B$.

$w \models K \mathsf{\ says\ } A$ iff either $w \in F$ or for all w', w'', $w \leq w' \sqsubseteq_K w''$ implies $w'' \models A$.

We say that a principal K validates A in model M (written $M \models^K A$) if for each world $w \in W^K$ in M, it is the case that $w \models A$. The Kripke semantics defined above are sound and complete in the following sense.

Theorem 4.3 (Soundness and Completeness)**.** *$\cdot \xrightarrow{K} A$ in the sequent calculus if and only if for each Kripke model M, $M \models^K A$.*

Soundness ("only if" direction) follows by an induction on the given sequent calculus proof. We must generalize the statement a little to allow non-empty hypotheses. See Appendix D.1 for details. The proof of completeness ("if" direction) uses a canonical model construction, which we describe next.

4.1 Canonical Kripke Model and Completeness

We describe a canonical Kripke model for DTL_0 that satisfies the following property: for each K and A, if $\cdot \stackrel{K}{\not\longrightarrow} A$, then there is a world $w \in W^K$ such that $w \not\models A$. From this property, it follows immediately that satisfaction in Kripke models is complete with respect to the sequent calculus in the sense of Theorem 4.3. Our construction of the canonical model generalizes Alechina et al's construction of canonical models for constructive S4 [7]. Before defining the canonical Kripke model, we make some preliminary definitions.

Definition 4.4 (Theory)**.** A theory is a tuple (Γ, S), where Γ is a set of formulas, and S is a set of principals.

Definition 4.5 (Filter)**.** A set S of principals is called a filter if there exists a principal K such that $S = \{K' \mid K' \succeq K\}$. Note that by definition, a filter always has a minimum element (K), and a maximum element (ℓ). In particular, a filter can never be the empty set.

Definition 4.6 (Prime Theory)**.** We call a theory (Γ, S) prime if the following hold:

1. (Prin-closure) S is a filter.
2. (Fact-closure) Γ is closed under $\xrightarrow{K}$ for each $K \in S$, i.e., for each $K \in S$, $\Gamma \xrightarrow{K} A$ implies $A \in \Gamma$.[5]

[5] If Γ is an infinite set, then $\Gamma \xrightarrow{K} A$ means that there is a finite subset Γ' of Γ such that $\Gamma' \xrightarrow{K} A$.

3. (Primality) If $A \vee B \in \Gamma$, then either $A \in \Gamma$ or $B \in \Gamma$.

We take as worlds of our canonical model all prime theories (Γ, S). The key property that we ensure in our construction is that $(\Gamma, S) \models A$ iff $A \in \Gamma$. Then, our proof of completeness is as follows. Suppose $\cdot \stackrel{K}{\not\rightarrow} A$. We define $S_K = \{K' \mid K' \succeq K\}$ and construct a prime theory (Γ^*, S_K) such that $A \notin \Gamma^*$. By the key property, $(\Gamma^*, S_K) \not\models A$. This completes the proof. There are three essential steps in this proof:

(a) Defining the canonical model whose worlds are prime theories

(b) Showing that $(\Gamma, S) \models A$ iff $A \in \Gamma$

(c) Showing that we can construct the prime theory (Γ^*, S_K) such that $A \notin \Gamma^*$

We start by defining the canonical model.

Definition 4.7 (Canonical Kripke Model)**.** The canonical Kripke model for DTL_0 is the tuple $(W, \theta, \leq, (\sqsubseteq_K)^{K \in \mathtt{Prin}}, \rho, F)$, where

W is the set of all prime theories (Γ, S)

$\theta(\Gamma, S) = S$

$(\Gamma, S) \leq (\Gamma', S')$ iff $\Gamma \subseteq \Gamma'$ and $S \subseteq S'$

$(\Gamma, S) \sqsubseteq_K (\Gamma', S')$ iff $K \in S'$ and for each A, K says $A \in \Gamma$ implies $A \in \Gamma'$

$P \in \rho(\Gamma, S)$ iff $P \in \Gamma$

$(\Gamma, S) \in F$ iff $\perp \in \Gamma$

The following lemma shows that the above definition actually describes a Kripke model for DTL_0. (Detailed proofs of all lemmas in this section are in Appendix D.2.)

Lemma 4.8 (Canonical Model)**.** *The model constructed in Definition 4.7 is a Kripke model for DTL_0, i.e., it satisfies all conditions of Definition 4.1.*

Proof. We may directly verify each condition from Definition 4.1. □

Next we introduce a notion of consistency for theories with respect to formulas. This notion is needed to establish steps (b) and (c) in our proof of completeness. For a filter S we say that the theory (Γ, S) is A consistent, if $\Gamma \stackrel{K}{\not\rightarrow} A$ for any $K \in S$. The following critical lemma states that any A consistent theory can be *extended* to an A consistent prime theory.

Lemma 4.9 (Consistent Extensions)**.** *Let (Γ, S) be an A consistent theory. Then there is an A consistent prime theory (Γ^*, S) such that $\Gamma \subseteq \Gamma^*$.*

Proof. By a straightforward application of Zorn's Lemma. □

At this point, we can prove the central property of our canonical model, namely that $(\Gamma, S) \models A$ iff $A \in \Gamma$.

Lemma 4.10 (Satisfaction)**.** *For each formula A, and each prime theory (Γ, S), it is the case that $(\Gamma, S) \models A$ in the canonical model iff $A \in \Gamma$.*

Proof. By induction on A. The cases $A = B \supset C$ and $A = K \text{ says } B$ require Lemma 4.9. □

Finally, we prove completeness by combining Lemmas 4.9 and 4.10.

Theorem 4.11 (Completeness)**.** *Suppose $\cdot \not\xrightarrow{K} A$. Then there is a world w in the canonical model such that $K \in \theta(w)$ and $w \not\models A$.*

Proof. Let $S_K = \{K' \mid K' \succeq K\}$. Since $\cdot \not\xrightarrow{K} A$, the theory $(\cdot, S_K)$ is A consistent. By Lemma 4.9, there is an A consistent prime theory (Γ^*, S_K). Take $w = (\Gamma^*, S_K)$. Clearly, $K \in \theta(w)$ and $A \notin \Gamma^*$. Using the latter fact and Lemma 4.10, $(\Gamma^*, S_K) \not\models A$, as required. □

Theorem 4.3 follows as an easy corollary to this theorem.

5 Connections to Other Logics

Having described both the proof-theory and the semantics of DTL_0, we study connections between DTL_0 and other logics, including some authorization logics. Our technical approach is based on sound and complete translations between the logics. The purpose of studying these connections is two-fold. First, we wish to understand DTL_0 better through these translations. Second, through translations from existing authorization logics to DTL_0, we seek to argue that DTL_0 is at least as expressive as each of them. In future, we would also like to use these (or similar) translations to try to develop a single framework for combining policies written in different logics.

We start by observing in Section 5.1 that DTL_0 generalizes the modal logic constructive S4 or CS4 (without $\Diamond$) [7, 13, 46] in the following sense: the trivial embedding from CS4 to DTL_0 that maps $\Box A$ to $\ell \text{ says } A$, and every other connective to itself is sound and complete. At the same time, DTL_0 is quite distinct from another, rather obvious generalization of constructive S4: the constructive multi-modal S4 that keeps modalities independent of each other (called CS4^m here). For example, the latter logic validates $(K \text{ says } K' \text{ says } A) \supset K' \text{ says } A$, which DTL_0 does not. The question then is whether there is a connection between DTL_0 and CS4^m. We show that there is an easy sound and complete embedding of DTL_0 into CS4^m. We do not know whether an embedding exists in the other direction.

Next, we examine connections to existing authorization logics. Recently, a number of authorization logics have been proposed [3, 25, 31–33] that treat $K \text{ says } \cdot$ as a modality from lax logic [12, 27, 28]. Although these logics differ in constructs other than says, each of them treats the modality $K \text{ says } \cdot$ in the same way. In Section 5.2, we describe a propositional core that is common to all these authorization logics, and show that it can be translated to DTL_0. By considering the degenerate case where the source of the translation has only one modality, we obtain a translation from lax logic to DTL_0. (A different but related translation from lax logic to S4 appeared in prior work [31]).

The earliest authorization logics [6, 39] treated $K \text{ says } \cdot$ as the weakest normal $\Box$ modality, i.e., the necessitation modality from the modal logic K. Although these logics

were classical, the interpretation of K says $\cdot$ as a weak normal modality may be useful even in intuitionistic authorization logics. In Section 5.3, we describe a propositional intuitionistic authorization logic with weak normal modalities, and translate it to DTL_0. In Section 5.4 we present a translation from a *language* for writing authorization policies, namely Soutei [47], to DTL_0.

Finally, in Section 5.5 we use a suggestion by Abadi [2] and strengthen DTL_0 to a logic which includes the axiom $(K \text{ says } A) \supset K' \text{ says } K \text{ says } A$. We describe the proof-theory of the logic briefly and show that it admits a simple translation to DTL_0. The axiom $(K \text{ says } A) \supset K' \text{ says } K \text{ says } A$ is stronger than axiom (4) of DTL_0, and, as observed by Abadi [2], seems to capture the essence of says in some languages like Soutei and Binder [24]. Our translations formalize this observation. We also present a sound and complete translation from this logic to CS4.

5.1 Connection to CS4

Constructive S4, or CS4 for short, is an intuitionistic version of the modal logic S4 [7, 13, 46]. As usual, it contains the modalities of necessity ($\Box A$) and possibility ($\Diamond A$). We are concerned here with propositional CS4 without $\Diamond$. A Hilbert style proof system for this logic consists of any axiomatization of intuitionistic propositional logic, and the following rules and axioms for $\Box A$ [7].

$$\frac{\vdash A}{\vdash \Box A} \qquad \text{(nec)}$$

$$\vdash (\Box(A \supset B)) \supset ((\Box A) \supset (\Box B)) \qquad \text{(K)}$$
$$\vdash (\Box A) \supset \Box\Box A \qquad \text{(4)}$$
$$\vdash (\Box A) \supset A \qquad \text{(T)}$$

$\mathbf{DTL}_0$ as a Generalization of CS4. An obvious translation from CS4 to DTL_0 is to map $\Box A$ to ℓ says A and all other connectives to themselves. Remarkably, this simple translation is both sound and complete. Another way to look at this translation is to say that in the degenerate case where there is only one principal (say ℓ) in DTL_0, the sole modality ℓ says A behaves exactly like the necessitation modality $\Box A$ from CS4. In fact, in this degenerate case the natural deduction system for DTL_0 (Figure 1) reduces to the judgmental natural deduction system for CS4 developed by Pfenning and Davies [46]. Similarly, the sequent calculus (Figure 4) reduces to a corresponding calculus for CS4 (e.g., [32]). Moreover, the Kripke semantics of DTL_0 reduce to those of CS4 described by Alechina et al. [7] (without $\Diamond$), with the minor difference that our treatment of falsehood uses fallible worlds explicitly. The following theorem is straightforward.

Theorem 5.1. *In the special case where there is only one principal ℓ in DTL_0, the following are equivalent:*

1. *$\vdash A$ treating ℓ says $\cdot$ as a CS4 $\Box$ modality.*
2. *$\vdash^{\ell} A$ in the natural deduction system of Figure 1.*

Proof. First we observe that in the natural deduction system of Figure 1, the contexts associated with hypothetical judgments become meaningless if there is only one principal ℓ. This is because the only place where contexts are used is the premise $K' \succeq K$ of the rule (claims). With only one possible principal, $K' \succeq K$ is always true by the fact that $\succeq$ is a pre-order. Next we observe that with contexts erased from the hypothetical judgments, the natural deduction system for DTL_0 becomes the same as the judgmental natural deduction system of CS4 [46], taking ℓ claims A to be the judgment called A valid in CS4, and taking ℓ says A to be $\Box A$ in CS4. It follows then that any theorem validated in DTL_0 for this degenerate case is also validated in CS4, and viceversa. □

The above theorem shows that DTL_0 generalizes CS4. A different generalization of CS4 may be obtained by taking several necessitation modalities that are independent of each other. We call this logic $\mathrm{CS4}^m$. In the following we briefly describe $\mathrm{CS4}^m$, observe that it is different from DTL_0, and present a sound and complete translation from DTL_0 to $\mathrm{CS4}^m$.

$\mathbf{CS4}^m$. The logic $\mathrm{CS4}^m$ extends intuitionistic propositional logic with one necessitation modality for each principal K, written $\Box_K A$. As in DTL_0, we assume a pre-order $\succeq$ between principals, and also that there is a maximum principal ℓ. The following rules and axioms apply to $\Box_K A$.

$$\frac{\vdash A}{\vdash \Box_K A} \qquad \text{(nec)}$$

$$\vdash (\Box_K(A \supset B)) \supset ((\Box_K A) \supset (\Box_K B)) \qquad \text{(K)}$$
$$\vdash (\Box_K A) \supset \Box_K \Box_K A \qquad \text{(4)}$$
$$\vdash (\Box_K A) \supset A \qquad \text{(T)}$$
$$\vdash (\Box_K A) \supset \Box_{K'} A \text{ if } K \succeq K' \qquad \text{(S)}$$

(nec)–(T) mean that each modality $\Box_K$ behaves like a CS4 necessitation modality. Axiom (S) incorporates the pre-order $\succeq$ into logical reasoning. A simpler logic, similar to $\mathrm{CS4}^m$, without the pre-order $\succeq$ has been studied in the past to model *knowledge* in authorization policies [32].

Relation between $\mathbf{CS4}^m$ and $\mathbf{DTL}_0$. It is easy to see that the modality $\Box_K A$ in $\mathrm{CS4}^m$ is quite different from K says A in DTL_0. For example, $(\Box_K \Box_{K'} A) \supset \Box_{K'} A$ by axiom (T), but K says K' says A does not always imply K' says A in DTL_0. However, there is a simple sound and complete translation from DTL_0 to $\mathrm{CS4}^m$. Assume that both the set of principals and the ordering $\succeq$ on them are the same in DTL_0 and $\mathrm{CS4}^m$. Further assume that for each principal K, there is a distinct atomic formula in $\mathrm{CS4}^m$, also written K. Assuming that these atomic formulas are disjoint from the usual atomic formulas P, we define a translation $\ulcorner\cdot\urcorner$ from formulas of DTL_0 to formulas of $\mathrm{CS4}^m$ as

follows.

$$\begin{array}{rcl}
\ulcorner P \urcorner & = & P \\
\ulcorner A \wedge B \urcorner & = & \ulcorner A \urcorner \wedge \ulcorner B \urcorner \\
\ulcorner A \vee B \urcorner & = & \ulcorner A \urcorner \vee \ulcorner B \urcorner \\
\ulcorner A \supset B \urcorner & = & \ulcorner A \urcorner \supset \ulcorner B \urcorner \\
\ulcorner \top \urcorner & = & \top \\
\ulcorner \bot \urcorner & = & \bot \\
\ulcorner K \text{ says } A \urcorner & = & \Box_K (K \supset \ulcorner A \urcorner)
\end{array}$$

The important part of the translation is the mapping of K says A to $\Box_K(K \supset \ulcorner A \urcorner)$. The formula K on the left of the implication acts as a "guard" on $\ulcorner A \urcorner$, and recovers the effect of the context associated with hypothetical judgments in DTL$_0$: $\ulcorner A \urcorner$ can be obtained from $K \supset \ulcorner A \urcorner$ only if K is true. By design, our translation ensures that K is true if and only if we are reasoning in a context weaker than K.

Define the set of formulas $\mathcal{O} = \{\Box_\ell(K \supset K') \mid K' \succeq K\}$. $\mathcal{O}$ captures the pre-order $\succeq$ between principals as implications between the representations of principals as atomic formulas. The following theorem states the correctness property for the translation. (We abuse notation slightly and use $\mathcal{O}$ to also represent the formula obtained by taking the conjunction of all formulas in the set $\mathcal{O}$.)

Theorem 5.2 (Correctness). *$\cdot \xrightarrow{K} A$ in DTL$_0$ if and only if $\vdash \mathcal{O} \supset (K \supset \ulcorner A \urcorner)$ in CS4^m.*

Proof. A detailed proof of this theorem is in Appendix E. A brief outline of the proof is as follows. Soundness ("only if" direction) is established by an induction on proofs in DTL$_0$. For convenience, we induct on proofs in the axiomatic system, but we could also have inducted either on proofs in the sequent calculus or on proofs in the natural deduction system.

Completeness ("if" direction) is established through a semantic argument. First we define an interpretation of CS4^m formulas in Kripke models of DTL$_0$, and show that the interpretation is sound. This works because the two logics CS4^m and DTL$_0$ are similar. Next, we show that for each DTL$_0$ formula A, $\models A$ if and only if $\models \ulcorner A \urcorner$. Then, completeness of the translation follows from completeness of DTL$_0$ with respect to its Kripke models (Theorem 4.3). Alternatively, we could have used a purely syntactic argument to establish completeness, as we do for all later translations in this section. However, doing so would require that we also develop a sequent calculus for CS4^m. Although this is straightforward, it was tempting to avoid the extra work, and to use the results already developed for the Kripke semantics of DTL$_0$. Indeed, this turned out to be a good choice since the semantic proof is both short and easy. $\Box$

5.2 Translation from an Authorization Logic with Lax Modalities

In the recent past, a number of authorization logics have been proposed [3, 25, 31–33] that treat K says $\cdot$ as a modality from lax logic [12, 27, 28]. The well studied semantics and proof-theory of lax logic (e.g., [7, 27, 38, 46]) generalize to these authorization logics [3, 31, 33]. Also, useful meta-theoretic properties such as non-interference can be established readily for such authorization logics [3, 33]. Owing to these merits, a number of proposals have used authorization logics based on lax modalities (e.g., [29, 40, 50]),

in particular the interpretation of the Dependency Core Calculus [5] as an authorization logic [3].

Although authorization logics that interpret $K\ \mathsf{says}\ \cdot$ as a lax modality differ widely in the connectives and constructs allowed, an intuitionistic propositional fragment is common to all of them. We call this common fragment ICL, borrowing the name from earlier work [31], and show that it can be translated to DTL_0.

ICL. The logic ICL extends intuitionistic propositional logic with a principal-indexed modality $K\ \mathsf{says}\ \cdot$, which satisfies the following axioms.

$$\begin{array}{ll} \vdash A \supset (K\ \mathsf{says}\ A) & \text{(unit)} \\ \vdash (K\ \mathsf{says}\ (A \supset B)) \supset ((K\ \mathsf{says}\ A) \supset K\ \mathsf{says}\ B) & \text{(K)} \\ \vdash (K\ \mathsf{says}\ K\ \mathsf{says}\ A) \supset K\ \mathsf{says}\ A & \text{(C4)} \end{array}$$

(unit) is the characterizing axiom of ICL. It means that any true formula is believed by all principals, thus making truth irrefutable by principals. (unit) also subsumes the (nec) rule. (C4), together with (unit), forces $(K\ \mathsf{says}\ A) \equiv (K\ \mathsf{says}\ K\ \mathsf{says}\ A)$. With these axioms, $K\ \mathsf{says}\ \cdot$ behaves exactly like the lax modality. Unlike DTL_0, there is no order between principals in ICL. A detailed description of the proof-theory and semantics of ICL may be found in earlier work [3, 31, 33].

Translation from ICL to $\mathbf{DTL}_0$. Let us assume that all principals in ICL also exist in DTL_0, and that these principals are unrelated to each other in the order $\succeq$. Then, we may translate ICL to DTL_0 as follows. (We remind the reader that the connectives $\mathsf{global}\ A$ and $K\ \mathsf{publ}\ A$ were defined in Section 2.)

$$\begin{array}{rcccc} \ulcorner P \urcorner & = & \mathsf{global}\ P & & \\ \ulcorner A \wedge B \urcorner & = & \ulcorner A \urcorner \wedge \ulcorner B \urcorner & & \\ \ulcorner A \vee B \urcorner & = & \ulcorner A \urcorner \vee \ulcorner B \urcorner & & \\ \ulcorner \top \urcorner & = & \top & & \\ \ulcorner \bot \urcorner & = & \bot & & \\ \ulcorner A \supset B \urcorner & = & \mathsf{global}\ (\ulcorner A \urcorner \supset \ulcorner B \urcorner) & & \\ \ulcorner K\ \mathsf{says}\ A \urcorner & = & \mathsf{global}\ (K\ \mathsf{says}\ \ulcorner A \urcorner) & = & K\ \mathsf{publ}\ \ulcorner A \urcorner \end{array}$$

The basic idea of our translation is to prefix some of the connectives with the defined modality global, so that for each A, it is the case that $\ulcorner A \urcorner \supset \mathsf{global}\ \ulcorner A \urcorner$.[6] Then by the properties of global listed above, it follows that $\ulcorner A \urcorner \supset \ulcorner K\ \mathsf{says}\ A \urcorner$. This captures the effect of the axiom (unit) in the translation. Soundness of the remaining axioms of ICL is straightforward. Interestingly, this translation is also complete. The following theorem states this formally.

Theorem 5.3 (Correctness)**.** *$\vdash A$ in ICL if and only if $\cdot \xrightarrow{\ell} \ulcorner A \urcorner$ in DTL_0.*

Proof. Appendix F contains a complete proof of this theorem. Soundness ("only if" direction) is readily established by induction on the derivation of $\vdash A$. Completeness ("if" direction) is established using a simulation technique based in sequent calculi.

[6] Our translation is inspired by, and resembles Gödel's translation from intuitionistic logic to classical modal S4, where a $\Box$ is put before each connective [36].

First, we syntactically characterize those DTL_0 sequents that may occur in the proof of a formula obtained by translation. These sequents are called *regular*. This characterization relies on the subformula property of DTL_0's sequent calculus. Next, we define an inverse translation from regular sequents to sequents of ICL, and use induction on sequent calculus derivations to prove a simulation result: any DTL_0 proof ending in a regular sequent can be simulated in ICL. From this fact completeness follows immediately. This method scales quite well to translations between other logics as long as the target of the translation has a sequent calculus with the subformula property. In particular, we use the method to prove completeness of all translations described later in this section. For some other applications of the method, see prior work [31, 34]. $\square$

5.3 Translation from an Authorization Logic with Weak Normal Modalities

A necessitation modality is called normal if it satisfies the rule (nec) and the axiom (K). In the earliest authorization logics [6, 39], each modality $K \textsf{ says } \cdot$ was treated as the weakest possible necessitation modality, i.e., a modality that admits (nec), (K), and their consequences only. Although these early logics were classical, the treatment of $K \textsf{ says } \cdot$ as the weakest normal necessitation modality may be interesting in an intuitionistic authorization logic as well. In the following we present an intuitionistic authorization logic that admits only (nec) and (K), and describe a sound and complete translation from it to DTL_0. We call the new logic IIK (Indexed Intuitionistic K).

IIK. The logic IIK is an extension of intuitionistic propositional logic with formulas $K \textsf{ says } A$ that satisfy the following rule and axiom:

$$\frac{\vdash A}{\vdash K \textsf{ says } A} \qquad \text{(nec)}$$

$$\vdash (K \textsf{ says } (A \supset B)) \supset ((K \textsf{ says } A) \supset (K \textsf{ says } B)) \qquad \text{(K)}$$

These two together imply that for each K, $K \textsf{ says } \cdot$ is the weakest normal necessitation modality. As in ICL, there is no order between principals in IIK.

Translation to DTL_0. We assume the existence of a distinguished principal d in DTL_0 that is distinct from ℓ and all principals in IIK and that is *unrelated* in the order $\succeq$ to all principals except itself and ℓ. Then we define a translation from IIK to DTL_0 as follows.

$$\begin{array}{rcl}
\ulcorner P \urcorner & = & P \\
\ulcorner A \wedge B \urcorner & = & \ulcorner A \urcorner \wedge \ulcorner B \urcorner \\
\ulcorner A \vee B \urcorner & = & \ulcorner A \urcorner \vee \ulcorner B \urcorner \\
\ulcorner \top \urcorner & = & \top \\
\ulcorner \bot \urcorner & = & \bot \\
\ulcorner A \supset B \urcorner & = & \ulcorner A \urcorner \supset \ulcorner B \urcorner \\
\ulcorner K \textsf{ says } A \urcorner & = & d \textsf{ says } K \textsf{ says } \ulcorner A \urcorner
\end{array}$$

This translation maps all connectives except $\textsf{says}$ to themselves, and maps $K \textsf{ says } A$ to $d \textsf{ says } K \textsf{ says } A$. It is easy to check that the compound connective $d \textsf{ says } K \textsf{ says } \cdot$ admits

both the rule (nec) and the axiom (K) in DTL_0, but not the axioms (C) and (4). This ensures that the translation is both sound and complete. The following theorem states this formally.

Theorem 5.4 (Correctness). *$\vdash A$ in IIK if and only if $\cdot \xrightarrow{\ell} \ulcorner A \urcorner$ in DTL_0.*

Proof. See Appendix G. □

5.4 Translation from Soutei

Soutei is a trust management system for enforcing authorization policies [47]. Soutei's language for writing authorization policies has a syntax similar to that of authorization logics. For instance, there is a construct K says A, and there are conditionals similar to logical implication. Like other trust management systems (e.g., [16–18]), Soutei is query based: other programs provide authorization policies, and ask whether specific authorizations follow from them. Soutei's mechanism for query evaluation is based on ideas from logic programming. There are fixed inference rules that constitute a decision procedure similar to backchaining. Although the policy language is first-order, we consider here only a simplified propositional fragment of the language and show that it can be translated into DTL_0. For the lack of a better name, we call Soutei's policy language SL.

SL. Soutei's policy language is based on another language for writing authorization policies, Binder [24]. Policy statements (called clauses) are divided into disjoint sets called *assertions*. Each assertion has a name, which is analogous to a principal in authorization logics. If $A_1, \ldots, A_n$ are the clauses in an assertion named K, then we may think of the whole assertion as the hypothesis K publ $A_1, \ldots, K$ publ A_n. A simplified and abstracted syntax for SL without first-order quantification is shown below.[7]

$$
\begin{array}{llcl}
\text{Principals or names} & K & & \\
\text{Atomic Formulas} & P & & \\
\text{Goals} & G & ::= & P \mid K\ \mathsf{says}\ P \\
\text{Clauses} & A & ::= & P \leftarrow G_1, \ldots, G_n \\
\text{Assertions} & \Delta & ::= & A_1, \ldots, A_n \\
\text{Named assertions} & N & ::= & K : \Delta \\
\text{Hypotheses} & \Gamma & ::= & N_1, \ldots, N_k \\
\text{Queries} & q & ::= & \Delta \vdash_\Gamma G
\end{array}
$$

Policy statements are represented as clauses that have the form $P \leftarrow G_1, \ldots, G_n$, where P is an atomic formula and each G_i is either an atomic formula or has the form K says P. As usual, the entire clause means that P holds if each of $G_1, \ldots, G_n$ holds. n may be zero, in which case P is a fact. An assertion Δ is a set of clauses. A *named assertion* is a pair $K : \Delta$ containing an assertion and a principal. The principal is a name for the assertion, and may represent a physical domain (such as a computer or a user) inside which policies contained in the assertion hold. The set of all named assertions is called the hypothesis Γ. It is assumed implicitly that the names of all assertions in Γ are distinct. Queries are evaluated relative to the hypothesis Γ and an assertion Δ

[7] We change Soutei's original notation to make it consistent with our own notation. We also simplify the evaluation rules slightly, without affecting their consequences.

containing clauses which are valid at the point of evaluation. As evaluation of a query proceeds, Δ may change, but Γ remains fixed. Evaluation of queries is goal directed, and uses the following two rules:

$$\frac{(P \leftarrow G_1, \ldots, G_n) \in \Delta \qquad (\Delta \vdash_\Gamma G_i)_{i \in \{1,\ldots,n\}}}{\Delta \vdash_\Gamma P}\text{bc} \qquad \frac{(K : \Delta) \in \Gamma \qquad \Delta \vdash_\Gamma P}{\Delta' \vdash_\Gamma K \ \mathsf{says}\ P}\text{says}$$

The rule (bc) means that P holds if there is a clause $P \leftarrow G_1, \ldots, G_n$ in the valid assertion, and each G_i holds. This is the standard backchaining rule for logic programs. The rule (says) means that K says P is true if in the assertion Δ named K, P is true. At the top level, evaluation of a query begins in an assertion that has the distinguished name `system`.

Translation from SL to DTL$_0$. Let us assume that DTL$_0$ contains all principals in SL, including the distinguished principal `system` and in addition contains the principal ℓ. We further assume that principals are only related to themselves and ℓ in the order $\succeq$. Then, we define the following translation from SL to DTL$_0$.

Goals	G	$\ulcorner P \urcorner$	$=$	P
		$\ulcorner K \ \mathsf{says}\ P \urcorner$	$=$	$K \ \mathsf{publ}\ P$
Clauses	A	$\ulcorner P \leftarrow G_1, \ldots, G_n \urcorner$	$=$	$(\ulcorner G_1 \urcorner \wedge \ldots \wedge \ulcorner G_n \urcorner) \supset P$
Assertions	Δ	$\ulcorner A_1, \ldots, A_n \urcorner$	$=$	$\ulcorner A_1 \urcorner, \ldots, \ulcorner A_n \urcorner$
Named assertions	N	$\ulcorner K : A_1, \ldots, A_n \urcorner$	$=$	$K \ \mathsf{publ}\ \ulcorner A_1 \urcorner, \ldots, K \ \mathsf{publ}\ \ulcorner A_n \urcorner$
Hypotheses	Γ	$\ulcorner N_1, \ldots, N_k \urcorner$	$=$	$\ulcorner N_1 \urcorner, \ldots, \ulcorner N_k \urcorner$

The most significant part of the above translation is the use of the defined connective publ to translate goals of the form K says P as well as named assertions. Since named assertions are always available in SL (hypothesis never change when we evaluate a query), it is essential that the same be true in the image of the translation. Using publ in the translation of named assertions ensures this. The use of publ in the translation of goals of the form K says P is optional; we could also have translated goals K says P to K says P, without affecting the correctness of the translation, which is stated in the following theorem.

Theorem 5.5 (Correctness)**.** *Suppose* $(K : \Delta) \in \Gamma$*. Then* $\Delta \vdash_\Gamma G$ *in SL if and only if* $\ulcorner \Gamma \urcorner, \ulcorner \Delta \urcorner \xrightarrow{K} \ulcorner G \urcorner$ *in DTL*$_0$*.*

Proof. See Appendix H. □

5.5 "Binder" Logic and its Translation

In this section we consider a logic containing an axiom that is stronger than (4). We call this axiom (Bind) for reasons that will soon be clear.

$$(K \ \mathsf{says}\ A) \supset K' \ \mathsf{says}\ K \ \mathsf{says}\ A \quad \text{(Bind)}$$

The (Bind) axiom dates back to a survey of applications of logic in access control by Abadi [2]. In that paper, Abadi states that the (Bind) axiom is closely connected to the authorization language Binder [24]. (Recall from Section 5.4 that Binder is the precursor

to, and very similar to, Soutei's policy language SL.) The paper suggests that this axiom, together with some other basic modal axioms, is sufficient to justify Binder's rules. It follows from the connections between Binder and SL that the same set of axioms is also sufficient to justify SL's evaluation rules. It is not mentioned if Binder's evaluation rules are complete with respect to this axiom.

(Bind) is interesting from our perspective because replacing axiom (4) with (Bind) in DTL_0 results in a new authorization logic, which we call BL_0.[8] This logic is closely related to DTL_0 and also has very appealing proof-theoretic properties. First, says in this new logic behaves *exactly* like the defined connective publ in DTL_0. In fact we show that BL_0 may be embedded into DTL_0 by mapping $K \mathsf{\ says\ } A$ to $K \mathsf{\ publ\ } A$ and all other connectives to themselves. Second, we obtain a sequent calculus for BL_0 by making a small change to the sequent calculus of DTL_0. Third, we show that SL can be interpreted in BL_0 in a sound and complete manner through a translation that maps named assertions using says, thus formulating a variant of Abadi's observation as a concrete theorem. Fourth, we argue that ICL can be embedded in BL_0 in a sound and complete manner, using the translation described in Section 5.2. Finally, we adapt the translation from DTL_0 to $\mathrm{CS4}^m$ presented in Section 5.1 to obtain a translation from DTL_0 to CS4. Although we do not do so here, we also expect that there are sound and complete Kripke semantics for BL_0 that are very similar to (but simpler than) the Kripke semantics of DTL_0 (Section 4).

The Logic BL_0

BL_0 extends intuitionistic propositional logic with the modality $K \mathsf{\ says\ } A$ satisfying the following rules and axioms.

$$\frac{\vdash A}{\vdash K \mathsf{\ says\ } A} \qquad \text{(nec)}$$

$$\begin{array}{ll}
\vdash (K \mathsf{\ says\ } (A \supset B)) \supset ((K \mathsf{\ says\ } A) \supset (K \mathsf{\ says\ } B)) & \text{(K)} \\
\vdash (K \mathsf{\ says\ } A) \supset K' \mathsf{\ says\ } K \mathsf{\ says\ } A & \text{(Bind)} \\
\vdash K \mathsf{\ says\ } ((K \mathsf{\ says\ } A) \supset A) & \text{(C)}
\end{array}$$

The axiom (Bind) generalizes axiom (4) of DTL_0. Unlike DTL_0, we do not assume any order between principals (although such an extension is easily conceivable). A sequent calculus for BL_0 is shown in Figure 5. This sequent calculus is a modification of the sequent calculus for DTL_0 (Figure 4). The notation $\Gamma|$ in the rule (saysR) denotes the set containing "claims" of all principals.

$$\Gamma| \;=\; \{K \mathsf{\ claims\ } C \in \Gamma\}$$

This change in the restriction operator is sufficient to capture the generalization from axiom (4) to (Bind). Besides this difference, the rule (claims) is modified slightly to eliminate the order $\succeq$. Although we do not do so here, we may also prove admissibility of cut and identity theorems (Section 3.5) for BL_0. The following theorem shows that the sequent calculus and axiomatic system for BL_0 are equivalent.

[8] BL_0 is a fragment of a larger logic BL, just as DTL_0 is a fragment of DTL. BL stands for "Binder Logic", since its says modality is closely related to the policy language Binder [24].

$$\frac{P \text{ atomic}}{\Gamma, P \xrightarrow{K} P}\,\text{init} \qquad\qquad \frac{\Gamma, K\ \mathsf{claims}\ A, A \xrightarrow{K} C}{\Gamma, K\ \mathsf{claims}\ A \xrightarrow{K} C}\,\text{claims}$$

$$\frac{\Gamma| \xrightarrow{K} A}{\Gamma \xrightarrow{K'} K\ \mathsf{says}\ A}\,\mathsf{says}\text{R} \qquad\qquad \frac{\Gamma, K\ \mathsf{says}\ A, K\ \mathsf{claims}\ A \xrightarrow{K'} C}{\Gamma, K\ \mathsf{says}\ A \xrightarrow{K'} C}\,\mathsf{says}\text{L}$$

$$\frac{\Gamma \xrightarrow{K} A \qquad \Gamma \xrightarrow{K} B}{\Gamma \xrightarrow{K} A \wedge B}\,\wedge\text{R} \qquad\qquad \frac{\Gamma, A \wedge B, A, B \xrightarrow{K} C}{\Gamma, A \wedge B \xrightarrow{K} C}\,\wedge\text{L}$$

$$\frac{\Gamma \xrightarrow{K} A}{\Gamma \xrightarrow{K} A \vee B}\vee \text{R}_1 \qquad \frac{\Gamma \xrightarrow{K} B}{\Gamma \xrightarrow{K} A \vee B}\vee \text{R}_2 \qquad \frac{\Gamma, A \vee B, A \xrightarrow{K} C \qquad \Gamma, A \vee B, B \xrightarrow{K} C}{\Gamma, A \vee B \xrightarrow{K} C}\vee \text{L}$$

$$\frac{}{\Gamma \xrightarrow{K} \top}\,\top\text{R} \qquad\qquad \frac{}{\Gamma, \bot \xrightarrow{K} C}\,\bot\text{L}$$

$$\frac{\Gamma, A \xrightarrow{K} B}{\Gamma \xrightarrow{K} A \supset B}\,\supset\text{R} \qquad\qquad \frac{\Gamma, A \supset B \xrightarrow{K} A \qquad \Gamma, A \supset B, B \xrightarrow{K} C}{\Gamma, A \supset B \xrightarrow{K} C}\,\supset\text{L}$$

Figure 5: Sequent calculus for BL_0

Theorem 5.6 (Equivalence)**.** *$\cdot \xrightarrow{K} A$ in BL_0's sequent calculus if and only if $\vdash K\ \mathsf{says}\ A$ in BL_0's axiomatic system.*

Proof. See Appendix I. $\square$

Translation from BL_0 to DTL_0

The modality $K\ \mathsf{says}\ A$ in BL_0 behaves exactly like $K\ \mathsf{publ}\ A$ in DTL_0. In fact, we may translate BL_0 to DTL_0 by mapping says to publ as follows. (We assume that all principals in BL_0 are distinct from ℓ. Also, we use the notation $[\![\cdot]\!]$ for the translation instead of our usual notation $\ulcorner\cdot\urcorner$ to distinguish it from the translation from SL to BL_0 that follows.)

$$\begin{array}{ccc}
[\![P]\!] & = & P \\
[\![A \wedge B]\!] & = & [\![A]\!] \wedge [\![B]\!] \\
[\![A \vee B]\!] & = & [\![A]\!] \vee [\![B]\!] \\
[\![\top]\!] & = & \top \\
[\![\bot]\!] & = & \bot \\
[\![A \supset B]\!] & = & [\![A]\!] \supset [\![B]\!] \\
[\![K\ \mathsf{says}\ A]\!] & = & K\ \mathsf{publ}\ [\![A]\!]
\end{array}$$

This simple translation is both sound and complete, as the following theorem shows.

Theorem 5.7 (Correctness)**.** *$\Gamma \xrightarrow{K} A$ in BL_0's sequent calculus if and only if $[\![\Gamma]\!] \xrightarrow{K} [\![A]\!]$ in DTL_0's sequent calculus.*

Proof. See Appendix I. $\square$

Translation from ICL to BL$_0$

The translation from ICL to DTL$_0$ described in Section 5.2 is also sound and complete if the target logic is BL$_0$. Of course, we need to assume that BL$_0$ also has a local authority ℓ, and modify the rule (claims) accordingly. The reason that this works is as follows. Consider a formula A in ICL. We may translate this to DTL$_0$ in two ways. First, we may translate it directly using the translation from Section 5.2, obtaining $\ulcorner A \urcorner$. Second, we may translate it to BL$_0$ using the same translation and then further translate it to DTL$_0$ using the translation described above, obtaining $[\![\ulcorner A \urcorner]\!]$. Now it is easy to show by induction on A that in DTL$_0$, $\Gamma \xrightarrow{K} \ulcorner A \urcorner$ iff $\Gamma \xrightarrow{K} [\![\ulcorner A \urcorner]\!]$. Hence by Theorems 5.3 and 5.7 we get $\vdash A$ in ICL iff $\cdot \xrightarrow{\ell} \ulcorner A \urcorner$ in DTL$_0$ iff $\cdot \xrightarrow{\ell} [\![\ulcorner A \urcorner]\!]$ in DTL$_0$ iff $\cdot \xrightarrow{\ell} \ulcorner A \urcorner$ in BL$_0$.

Translation from SL to BL$_0$

We may translate SL to BL$_0$, much like we translated SL to DTL$_0$. Since the modality K says A in BL$_0$ behaves like K publ A in DTL$_0$, we use says in place of publ everywhere.

Goals	G	$\ulcorner P \urcorner$	$=$	P
		$\ulcorner K \mathsf{\ says\ } P \urcorner$	$=$	$K \mathsf{\ says\ } P$
Clauses	A	$\ulcorner P \leftarrow G_1, \ldots, G_n \urcorner$	$=$	$(\ulcorner G_1 \urcorner \wedge \ldots \wedge \ulcorner G_n \urcorner) \supset P$
Assertions	Δ	$\ulcorner A_1, \ldots, A_n \urcorner$	$=$	$\ulcorner A_1 \urcorner, \ldots, \ulcorner A_n \urcorner$
Named assertions	N	$\ulcorner K : A_1, \ldots, A_n \urcorner$	$=$	$K \mathsf{\ says\ } \ulcorner A_1 \urcorner, \ldots, K \mathsf{\ says\ } \ulcorner A_n \urcorner$
Hypotheses	Γ	$\ulcorner N_1, \ldots, N_k \urcorner$	$=$	$\ulcorner N_1 \urcorner, \ldots, \ulcorner N_k \urcorner$

Once again, this translation is sound and complete.

Theorem 5.8 (Correctness)**.** *Suppose* $(K : \Delta) \in \Gamma$. *Then* $\Delta \vdash_\Gamma G$ *in SL if and only if* $\ulcorner \Gamma \urcorner, \ulcorner \Delta \urcorner \xrightarrow{K} \ulcorner G \urcorner$ *in BL$_0$.*

Proof. Note that we have two translations from SL to DTL$_0$. First, we have the translation $\ulcorner \cdot \urcorner$ from Section 5.4. Second, we may compose the translations $\ulcorner \cdot \urcorner : SL \to \mathrm{BL}_0$ and $[\![\cdot]\!] : \mathrm{BL}_0 \to \mathrm{DTL}_0$ that are described above. It is very easy to check that the two translations are the same. ($[\![\cdot]\!]$ maps says to publ, which compensates the only difference between the translations from SL to BL$_0$ and SL to DTL$_0$.) Thus we get,

$$
\begin{array}{rcll}
\Delta \vdash_\Gamma G \text{ in SL} & \leftrightarrow & \ulcorner \Gamma \urcorner, \ulcorner \Delta \urcorner \xrightarrow{K} \ulcorner G \urcorner \text{ in DTL}_0 & \text{(Theorem 5.5)} \\
& = & [\![\ulcorner \Gamma \urcorner]\!], [\![\ulcorner \Delta \urcorner]\!] \xrightarrow{K} [\![\ulcorner G \urcorner]\!] \text{ in DTL}_0 & (\ulcorner \cdot \urcorner = \ulcorner \cdot \urcorner; [\![\cdot]\!]) \\
& \leftrightarrow & \ulcorner \Gamma \urcorner, \ulcorner \Delta \urcorner \xrightarrow{K} \ulcorner G \urcorner \text{ in BL}_0 & \text{(Theorem 5.7)}
\end{array}
$$

$\square$

Translation from BL$_0$ to CS4

Finally, we adapt the translation from DTL$_0$ to CS4^m (Section 5.1) to obtain a translation from BL$_0$ to CS4. The difference between the translations is that here we use $\Box$

instead of $\Box_K$ for translating K says A. This not only simplifies the translation, but also captures the effects of the axiom (Bind).

$$\begin{array}{rcl}
\ulcorner P \urcorner & = & P \\
\ulcorner A \wedge B \urcorner & = & \ulcorner A \urcorner \wedge \ulcorner B \urcorner \\
\ulcorner A \vee B \urcorner & = & \ulcorner A \urcorner \vee \ulcorner B \urcorner \\
\ulcorner A \supset B \urcorner & = & \ulcorner A \urcorner \supset \ulcorner B \urcorner \\
\ulcorner \top \urcorner & = & \top \\
\ulcorner \bot \urcorner & = & \bot \\
\ulcorner K \text{ says } A \urcorner & = & \Box(K \supset \ulcorner A \urcorner)
\end{array}$$

Theorem 5.9 (Correctness)**.** $\cdot \xrightarrow{K} A$ *in* BL_0 *if and only if* $\vdash K \supset \ulcorner A \urcorner$ *in CS4.*

Proof. See Appendix I. □

6 Related Work

Many authorization logics have been proposed in the past, all of which contain the modality K says A [2, 3, 8–10, 21, 23, 25, 31–33, 40, 41]. The axioms and rules used in these logics differ widely. The particular combination of rules used in DTL_0 appears to be novel. Perhaps most closely related to DTL_0 is a proposal by Abadi in a survey paper [2], where the axiom $(K \text{ says } A) \supset (K' \text{ says } K \text{ says } A)$ is suggested. says with this axiom behaves very much like the defined connective publ in DTL_0. In a recent paper, Abadi studies connections between many possible axiomatizations of says, as well as higher level policy constructs such as delegation and control [4].

Also related to DTL_0 is work on languages for authorization (e.g., [11, 24, 37, 47]), most notably the languages Soutei and Binder [24, 47]. Our use of the term "context" is borrowed from the latter. Binder was also one of the earliest languages to explicitly define a notion of exporting policies from one context to another, which is very similar to publication of policies illustrated in Section 2. The pre-order $\succeq$ on principals draws on ideas from the Dependency Core Calculus [3, 5], where the modal indices are elements of a lattice.

Our Kripke semantics, as well as the completeness proof, are based on those of Alechina et al's work [7] for constructive S4. View functions were used earlier by the author and Abadi to describe semantics of authorization logics with lax-like modalities [31]. Fallible worlds have been used in the past to explain intuitionistic logic [26, 49], and also in semantics of lax logic [27]. It also appears to us that DTL_0 may be closely related to intuitionistic hybrid logics, and especially to the work of Chadha and others [22], but further investigation is needed to make an explicit connection. The presentation of the sequent calculus for DTL_0 is inspired by Pfenning and Davies' work on constructive S4 [46], and more directly by earlier work of the author and others [32].

7 Conclusion

We have presented a new constructive authorization logic, which explicitly relativizes hypothetical reasoning to the policies of individual principals. We have described the

proof-theory and Kripke semantics of the logic. In ongoing work, we are considering extensions of the logic with first-order connectives, explicit time, and linearity to model other policy motifs. In a separate line of research, we are implementing a file system that uses this logic to represent policies of access control.

There are several other avenues for future work. For instance, there seem to be strong connections between DTL_0 and hybrid logics. A useful generalization of DTL_0 would be to internalize the pre-order $\succeq$ as a formula. Such an extension would allow us to model delegation, along lines of the "speaks for" connective present in some authorization logics [3, 6, 31, 39]. Although the proof-theory of such an extension is relatively straightforward, it would be interesting to see its effects on Kripke semantics.

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A β-reduction and η-expansion

This appendix lists all the β-reduction and η-expansion rules.

$\boxed{\beta\text{-reduction}}$

$$\frac{\Gamma \vdash^K t_1 : A \qquad \Gamma \vdash^K t_2 : B}{\Gamma \vdash^K \texttt{proj}_1\ \langle t_1, t_2\rangle \rightsquigarrow_\beta t_1 : A} \qquad \frac{\Gamma \vdash^K t_1 : A \qquad \Gamma \vdash^K t_2 : B}{\Gamma \vdash^K \texttt{proj}_2\ \langle t_1, t_2\rangle \rightsquigarrow_\beta t_2 : B}$$

$$\frac{\Gamma \vdash^K t : A \qquad \Gamma, x : A \vdash^K t_1 : C \qquad \Gamma, y : B \vdash^K t_2 : C}{\Gamma \vdash^K \texttt{case}(\texttt{inl}\ t, x.t_1, y.t_2) \rightsquigarrow_\beta [t/x]t_1 : C}$$

$$\frac{\Gamma \vdash^K t : B \qquad \Gamma, x : A \vdash^K t_1 : C \qquad \Gamma, y : B \vdash^K t_2 : C}{\Gamma \vdash^K \texttt{case}(\texttt{inr}\ t, x.t_1, y.t_2) \rightsquigarrow_\beta [t/y]t_2 : C}$$

$$\frac{\Gamma, x : A \vdash^K t_1 : B \qquad \Gamma \vdash^K t_2 : A}{\Gamma \vdash^K (\lambda x.t_1)\ t_2 \rightsquigarrow_\beta [t_2/x]t_1 : B} \qquad \frac{\Gamma|_K \vdash^K t_1 : A \qquad \Gamma, x : K\ \textsf{claims}\ A \vdash^{K'} t_2 : C}{\Gamma \vdash^{K'} (\{t_1\}_K \Rightarrow x.t_2) \rightsquigarrow_\beta [t_1/x]t_2 : C}$$

$\boxed{\eta\text{-expansion}}$

$$\frac{\Gamma \vdash^K t : A \wedge B}{\Gamma \vdash^K t \rightsquigarrow_\eta \langle \texttt{proj}_1\ t, \texttt{proj}_2\ t\rangle : A \wedge B} \qquad \frac{\Gamma \vdash^K t : A \vee B}{\Gamma \vdash^K t \rightsquigarrow_\eta \texttt{case}(t, x.\ \texttt{inl}\ x, y.\ \texttt{inr}\ y) : A \vee B}$$

$$\frac{\Gamma \vdash^K t : \top}{\Gamma \vdash^K t \rightsquigarrow_\eta \langle\rangle : \top} \qquad \frac{\Gamma \vdash^K t : \bot}{\Gamma \vdash^K t \rightsquigarrow_\eta \texttt{abort}\ t}$$

$$\frac{\Gamma \vdash^K t : A \supset B}{\Gamma \vdash^K t \rightsquigarrow_\eta \lambda x.(t\ x) : A \supset B}(x \notin \Gamma) \qquad \frac{\Gamma \vdash^{K'} t : K\ \textsf{says}\ A}{\Gamma \vdash^{K'} t \rightsquigarrow_\eta (t \Rightarrow x.\{x\}_K) : K\ \textsf{says}\ A}$$

$\boxed{\text{Congruence rules}}$

$$\frac{\Gamma|_K \vdash^K t \rightsquigarrow t' : A}{\Gamma \vdash^{K'} \{t\}_K \rightsquigarrow \{t'\}_K : K \text{ says } A}$$

$$\frac{\Gamma \vdash^{K'} t_1 \rightsquigarrow t_1' : K \text{ says } A \qquad \Gamma, x : K \text{ claims } A \vdash^{K'} t_2 : C}{\Gamma \vdash^{K'} (t_1 \Rightarrow x.t_2) \rightsquigarrow (t_1' \Rightarrow x.t_2) : C}$$

$$\frac{\Gamma \vdash^{K'} t_1 : K \text{ says } A \qquad \Gamma, x : K \text{ claims } A \vdash^{K'} t_2 \rightsquigarrow t_2' : C}{\Gamma \vdash^{K'} (t_1 \Rightarrow x.t_2) \rightsquigarrow (t_1 \Rightarrow x.t_2') : C}$$

$$\frac{\Gamma \vdash^K t_1 \rightsquigarrow t_1' : A \qquad \Gamma \vdash^K t_2 : B}{\Gamma \vdash^K \langle t_1, t_2 \rangle \rightsquigarrow \langle t_1', t_2 \rangle : A \wedge B} \qquad \frac{\Gamma \vdash^K t_1 : A \qquad \Gamma \vdash^K t_2 \rightsquigarrow t_2' : B}{\Gamma \vdash^K \langle t_1, t_2 \rangle \rightsquigarrow \langle t_1, t_2' \rangle : A \wedge B}$$

$$\frac{\Gamma \vdash^K t \rightsquigarrow t' : A \wedge B}{\Gamma \vdash^K (\texttt{proj}_1\ t) \rightsquigarrow (\texttt{proj}_1\ t') : A} \qquad \frac{\Gamma \vdash^K t \rightsquigarrow t' : A \wedge B}{\Gamma \vdash^K (\texttt{proj}_2\ t) \rightsquigarrow (\texttt{proj}_2\ t') : B}$$

$$\frac{\Gamma \vdash^K t \rightsquigarrow t' : A}{\Gamma \vdash^K (\texttt{inl}\ t) \rightsquigarrow (\texttt{inl}\ t') : A \vee B} \qquad \frac{\Gamma \vdash^K t \rightsquigarrow t' : B}{\Gamma \vdash^K (\texttt{inr}\ t) \rightsquigarrow (\texttt{inr}\ t') : A \vee B}$$

$$\frac{\Gamma \vdash^K t \rightsquigarrow t' : A \vee B \qquad \Gamma, x : A \vdash^K t_1 : C \qquad \Gamma, y : B \vdash^K t_2 : C}{\Gamma \vdash^K \texttt{case}(t, x.t_1, y.t_2) \rightsquigarrow \texttt{case}(t', x.t_1, y.t_2) : C}$$

$$\frac{\Gamma \vdash^K t : A \vee B \qquad \Gamma, x : A \vdash^K t_1 \rightsquigarrow t_1' : C \qquad \Gamma, y : B \vdash^K t_2 : C}{\Gamma \vdash^K \texttt{case}(t, x.t_1, y.t_2) \rightsquigarrow \texttt{case}(t, x.t_1', y.t_2) : C}$$

$$\frac{\Gamma \vdash^K t : A \vee B \qquad \Gamma, x : A \vdash^K t_1 : C \qquad \Gamma, y : B \vdash^K t_2 \rightsquigarrow t_2' : C}{\Gamma \vdash^K \texttt{case}(t, x.t_1, y.t_2) \rightsquigarrow \texttt{case}(t, x.t_1, y.t_2') : C}$$

$$\frac{\Gamma \vdash^K t \rightsquigarrow t' : \bot}{\Gamma \vdash^K (\texttt{abort}\ t) \rightsquigarrow (\texttt{abort}\ t') : C} \qquad \frac{\Gamma, x : A \vdash^K t \rightsquigarrow t' : B}{\Gamma \vdash^K \lambda x.t \rightsquigarrow \lambda x.t' : A \supset B}$$

$$\frac{\Gamma \vdash^K t_1 \rightsquigarrow t_1' : A \supset B \qquad \Gamma \vdash^K t_2 : A}{\Gamma \vdash^K t_1\ t_2 \rightsquigarrow t_1'\ t_2 : B} \qquad \frac{\Gamma \vdash^K t_1 : A \supset B \qquad \Gamma \vdash^K t_2 \rightsquigarrow t_2' : A}{\Gamma \vdash^K t_1\ t_2 \rightsquigarrow t_1\ t_2' : B}$$

B Properties of the Sequent Calculus (Section 3.5)

In this appendix, we describe proofs of Theorems from Section 3.5. We start with subsumption.

Theorem B.1 (Subsumption; Theorem 3.6)**.** $\Gamma \xrightarrow{K} A$ *and* $K \succeq K'$ *imply* $\Gamma \xrightarrow{K'} A$.

Proof. By induction on the given derivation of $\Gamma \xrightarrow{K} A$. We analyze cases of the last

rule in the derivation.

Case. $\dfrac{P \text{ atomic}}{\Gamma, P \xrightarrow{K} P}\text{init}$

1. $\Gamma, P \xrightarrow{K'} P$ (Rule (init))

Case. $\dfrac{\Gamma, K'' \mathsf{\ claims\ } A, A \xrightarrow{K} C \qquad K'' \succeq K}{\Gamma, K'' \mathsf{\ claims\ } A \xrightarrow{K} C}\text{claims}$

1. $\Gamma, K'' \mathsf{\ claims\ } A, A \xrightarrow{K'} C$ (i.h.)
2. $K'' \succeq K$ (Premise)
3. $K \succeq K'$ (Assumption)
4. $K'' \succeq K'$ (Transitivity 2, 3)
5. $\Gamma, K'' \mathsf{\ claims\ } A \xrightarrow{K'} C$ (Rule (claims) 1, 4)

Case. $\dfrac{\Gamma|_{K''} \xrightarrow{K''} A}{\Gamma \xrightarrow{K} K'' \mathsf{\ says\ } A}\mathsf{saysR}$

1. $\Gamma|_{K''} \xrightarrow{K''} A$ (Premise)
2. $\Gamma \xrightarrow{K'} K'' \mathsf{\ says\ } A$ (Rule (saysR))

Case. $\dfrac{\Gamma, K'' \mathsf{\ says\ } A, K'' \mathsf{\ claims\ } A \xrightarrow{K} C}{\Gamma, K'' \mathsf{\ says\ } A \xrightarrow{K} C}\mathsf{saysL}$

1. $\Gamma, K'' \mathsf{\ says\ } A, K'' \mathsf{\ claims\ } A \xrightarrow{K'} C$ (i.h.)
2. $\Gamma, K'' \mathsf{\ says\ } A \xrightarrow{K'} C$ (Rule (saysL))

Case. $\dfrac{\Gamma \xrightarrow{K} A \qquad \Gamma \xrightarrow{K} B}{\Gamma \xrightarrow{K} A \wedge B}\wedge\text{R}$

1. $\Gamma \xrightarrow{K'} A$ (i.h.)
2. $\Gamma \xrightarrow{K'} B$ (i.h.)
3. $\Gamma \xrightarrow{K'} A \wedge B$ (Rule ($\wedge$R) 1, 2)

Case. $\dfrac{\Gamma, A \wedge B, A, B \xrightarrow{K} C}{\Gamma, A \wedge B \xrightarrow{K} C}\wedge\text{L}$

1. $\Gamma, A \wedge B, A, B \xrightarrow{K'} C$ (i.h.)

2. $\Gamma, A \wedge B \xrightarrow{K'} C$ (Rule ($\wedge$L))

Case. $\dfrac{\Gamma \xrightarrow{K} A}{\Gamma \xrightarrow{K} A \vee B} \vee \mathrm{R}_1$

1. $\Gamma \xrightarrow{K'} A$ (i.h.)

2. $\Gamma \xrightarrow{K'} A \vee B$ (Rule ($\vee$ R_1))

Case. $\dfrac{\Gamma \xrightarrow{K} B}{\Gamma \xrightarrow{K} A \vee B} \vee \mathrm{R}_2$

1. $\Gamma \xrightarrow{K'} B$ (i.h.)

2. $\Gamma \xrightarrow{K'} A \vee B$ (Rule ($\vee$ R_2))

Case. $\dfrac{\Gamma, A \vee B, A \xrightarrow{K} C \qquad \Gamma, A \vee B, B \xrightarrow{K} C}{\Gamma, A \vee B \xrightarrow{K} C} \vee \mathrm{L}$

1. $\Gamma, A \vee B, A \xrightarrow{K'} C$ (i.h.)

2. $\Gamma, A \vee B, B \xrightarrow{K'} C$ (i.h.)

3. $\Gamma, A \vee B \xrightarrow{K'} C$ (Rule ($\vee$ L) 1, 2)

Case. $\dfrac{}{\Gamma \xrightarrow{K} \top} \top\mathrm{R}$

1. $\Gamma \xrightarrow{K'} \top$ (Rule ($\top$R))

Case. $\dfrac{}{\Gamma, \bot \xrightarrow{K} C} \bot\mathrm{L}$

1. $\Gamma, \bot \xrightarrow{K'} C$ (Rule ($\bot$L))

Case. $\dfrac{\Gamma, A \xrightarrow{K} B}{\Gamma \xrightarrow{K} A \supset B} \supset\mathrm{R}$

1. $\Gamma, A \xrightarrow{K'} B$ (i.h.)

2. $\Gamma \xrightarrow{K'} A \supset B$ (Rule ($\supset$R))

Case. $$\dfrac{\Gamma, A \supset B \xrightarrow{K} A \qquad \Gamma, A \supset B, B \xrightarrow{K} C}{\Gamma, A \supset B \xrightarrow{K} C}\supset\text{L}$$

1. $\Gamma, A \supset B \xrightarrow{K'} A$ (i.h.)
2. $\Gamma, A \supset B, B \xrightarrow{K'} C$ (i.h.)
3. $\Gamma, A \supset B \xrightarrow{K'} C$ (Rule ($\supset$L))

$\square$

Theorem B.2 (Admissibility of Cut; Theorem 3.7)**.** *The following cut principles hold for the sequent calculus of Figure 4.*

1. $\Gamma \xrightarrow{K} A$ *and* $\Gamma, A \xrightarrow{K} C$ *imply that* $\Gamma \xrightarrow{K} C$.
2. $\Gamma|_K \xrightarrow{K} A$ *and* $\Gamma, K\ \mathsf{claims}\ A \xrightarrow{K'} C$ *imply that* $\Gamma \xrightarrow{K'} C$.

Proof. We prove both statements simultaneously by lexicographic induction, first on the size of the cut judgment, and then on the size of the two given derivations, as in earlier work [45]. For the size of the cut judgment, we assume the strict order $(K\ \mathsf{says}\ A)\ \mathsf{true} > (K\ \mathsf{claims}\ A) > A\ \mathsf{true}$. We analyze cases on the last rules in the two given derivations, which we name $\mathcal{D}$ and $\mathcal{E}$ respectively. We classify all the rules into right and left. Right rules are (saysR), ($\wedge$R), ($\vee$ R$_1$), ($\vee$ R$_2$), ($\top$R), and ($\supset$R). The remaining rules, including (init), are left rules.

For proving (1), we first analyze three broad categories:

1. $\mathcal{E}$ ends in a right rule.
2. $\mathcal{E}$ ends in a left rule, and the cut is non-principal.
3. $\mathcal{D}$ ends in a left rule.

This leaves only the possibility where $\mathcal{E}$ ends in a left rule, $\mathcal{D}$ ends in a right rule, and the cut is principal. In this case we observe that the last rules in $\mathcal{D}$ and $\mathcal{E}$ must be right and left rules of the same connective. We call these cases principal cuts.

For proving (2), we analyze cases on the last rule in $\mathcal{E}$.

Proof of (1).

Cases where $\mathcal{E}$ ends in a right rule.

Case. $\mathcal{E} = \dfrac{\Gamma|_K \xrightarrow{K} B}{\Gamma, A \xrightarrow{K'} K\ \mathsf{says}\ B}\mathsf{says}\text{R}$ (Note: $A \notin \Gamma|_K$)

1. $\Gamma|_K \xrightarrow{K} B$ (Premise)
2. $\Gamma \xrightarrow{K'} K\ \mathsf{says}\ B$ (Rule (saysR))

Case. $\mathcal{E} = \dfrac{\Gamma, A \xrightarrow{K} C_1 \qquad \Gamma, A \xrightarrow{K} C_2}{\Gamma, A \xrightarrow{K} C_1 \wedge C_2} \wedge\mathrm{R}$

1. $\Gamma \xrightarrow{K} C_1$ (i.h. on $\mathcal{D}$ and 1st premise)
2. $\Gamma \xrightarrow{K} C_2$ (i.h. on $\mathcal{D}$ and 2nd premise)
3. $\Gamma \xrightarrow{K} C_1 \wedge C_2$ (Rule ($\wedge$R))

Case. $\mathcal{E} = \dfrac{\Gamma, A \xrightarrow{K} C_1}{\Gamma, A \xrightarrow{K} C_1 \vee C_2} \vee \mathrm{R}_1$

1. $\Gamma \xrightarrow{K} C_1$ (i.h. on $\mathcal{D}$ and premise)
2. $\Gamma \xrightarrow{K} C_1 \vee C_2$ (Rule ($\vee$ R_1))

Case. $\mathcal{E} = \dfrac{\Gamma, A \xrightarrow{K} C_2}{\Gamma, A \xrightarrow{K} C_1 \vee C_2} \vee \mathrm{R}_2$

1. $\Gamma \xrightarrow{K} C_2$ (i.h. on $\mathcal{D}$ and premise)
2. $\Gamma \xrightarrow{K} C_1 \vee C_2$ (Rule ($\vee$ R_2))

Case. $\mathcal{E} = \dfrac{}{\Gamma, A \xrightarrow{K} \top} \top\mathrm{R}$

1. $\Gamma \xrightarrow{K} \top$ (Rule ($\top$R))

Case. $\mathcal{E} = \dfrac{\Gamma, A, C_1 \xrightarrow{K} C_2}{\Gamma, A \xrightarrow{K} C_1 \supset C_2} \supset\mathrm{R}$

1. $\Gamma, C_1 \xrightarrow{K} C_2$ (i.h. on $\mathcal{D}$ and premise)
2. $\Gamma \xrightarrow{K} C_1 \supset C_2$ (Rule ($\supset$R))

Cases where $\mathcal{E}$ ends in a left rule and cut is not principal.

Case. $\mathcal{E} = \dfrac{P \text{ atomic}}{\Gamma, A, P \xrightarrow{K} P} \mathrm{init}$

1. $\Gamma, P \xrightarrow{K} P$ (Rule (init))

Case. $\mathcal{E} = \dfrac{\Gamma, A, K' \,\textsf{claims}\, B, B \xrightarrow{K} C \qquad K' \succeq K}{\Gamma, A, K' \,\textsf{claims}\, B \xrightarrow{K} C} \mathrm{claims}$

1. $\Gamma, K' \text{ claims } B, B \xrightarrow{K} C$ (i.h. on $\mathcal{D}$ and premise)

2. $\Gamma, K' \text{ claims } B \xrightarrow{K} C$ (Rule (claims))

Case. $\mathcal{E} = \dfrac{\Gamma, A, K' \text{ says } B, K' \text{ claims } B \xrightarrow{K} C}{\Gamma, A, K' \text{ says } B \xrightarrow{K} C}\text{saysL}$

1. $\Gamma, K' \text{ says } B, K' \text{ claims } B \xrightarrow{K} C$ (i.h. on $\mathcal{D}$ and premise)

2. $\Gamma, K' \text{ says } B \xrightarrow{K} C$ (Rule (saysL))

Case. $\mathcal{E} = \dfrac{\Gamma, A, B_1 \wedge B_2, B_1, B_2 \xrightarrow{K} C}{\Gamma, A, B_1 \wedge B_2 \xrightarrow{K} C}\wedge\text{L}$

1. $\Gamma, B_1 \wedge B_2, B_1, B_2 \xrightarrow{K} C$ (i.h. on $\mathcal{D}$ and premise)

2. $\Gamma, B_1 \wedge B_2 \xrightarrow{K} C$ (Rule ($\wedge$L))

Case. $\mathcal{E} = \dfrac{\Gamma, A, B_1 \vee B_2, B_1 \xrightarrow{K} C \qquad \Gamma, A, B_1 \vee B_2, B_2 \xrightarrow{K} C}{\Gamma, A, B_1 \vee B_2 \xrightarrow{K} C}\vee\text{L}$

1. $\Gamma, B_1 \vee B_2, B_1 \xrightarrow{K} C$ (i.h. on $\mathcal{D}$ and 1st premise)

2. $\Gamma, B_1 \vee B_2, B_2 \xrightarrow{K} C$ (i.h. on $\mathcal{D}$ and 2nd premise)

3. $\Gamma, B_1 \vee B_2 \xrightarrow{K} C$ (Rule ($\vee$L))

Case. $\mathcal{E} = \dfrac{}{\Gamma, A, \bot \xrightarrow{K} C}\bot\text{L}$

1. $\Gamma, \bot \xrightarrow{K} C$ (Rule ($\bot$L))

Case. $\mathcal{E} = \dfrac{\Gamma, A, B_1 \supset B_2 \xrightarrow{K} B_1 \qquad \Gamma, A, B_1 \supset B_2, B_2 \xrightarrow{K} C}{\Gamma, A, B_1 \supset B_2 \xrightarrow{K} C}\supset\text{L}$

1. $\Gamma, B_1 \supset B_2 \xrightarrow{K} B_1$ (i.h. on $\mathcal{D}$ and 1st premise)

2. $\Gamma, B_1 \supset B_2, B_2 \xrightarrow{K} C$ (i.h. on $\mathcal{D}$ and 2nd premise)

3. $\Gamma, B_1 \supset B_2 \xrightarrow{K} C$ (Rule ($\supset$L))

Cases where $\mathcal{D}$ ends in a left rule.

Case. $\mathcal{D} = \dfrac{A \text{ atomic}}{\Gamma, A \xrightarrow{K} A}\text{init}$

1. $\mathcal{E} :: \Gamma, A, A \xrightarrow{K} C$ (Assumption)

2. $\Gamma, A \xrightarrow{K} C$ (Strengthening)

Case. $\mathcal{D} = \dfrac{\Gamma, K' \text{ claims } B, B \xrightarrow{K} A \qquad K' \succeq K}{\Gamma, K' \text{ claims } B \xrightarrow{K} A}\text{claims}$

1. $\mathcal{E} :: \Gamma, K' \text{ claims } B, A \xrightarrow{K} C$ (Assumption)

2. $\Gamma, K' \text{ claims } B, B, A \xrightarrow{K} C$ (Weakening)

3. $\Gamma, K' \text{ claims } B, B \xrightarrow{K} C$ (i.h. on premise and 2)

4. $\Gamma, K' \text{ claims } B \xrightarrow{K} C$ (Rule (claims))

Case. $\mathcal{D} = \dfrac{\Gamma, K' \text{ says } B, K' \text{ claims } B \xrightarrow{K} A}{\Gamma, K' \text{ says } B \xrightarrow{K} A}\text{saysL}$

1. $\mathcal{E} :: \Gamma, K' \text{ says } B, A \xrightarrow{K} C$ (Assumption)

2. $\Gamma, K' \text{ says } B, K' \text{ claims } B, A \xrightarrow{K} C$ (Weakening)

3. $\Gamma, K' \text{ says } B, K' \text{ claims } B \xrightarrow{K} C$ (i.h. on premise and 2)

4. $\Gamma, K' \text{ says } B \xrightarrow{K} C$ (Rule (saysL))

Case. $\mathcal{D} = \dfrac{\Gamma, B_1 \wedge B_2, B_1, B_2 \xrightarrow{K} A}{\Gamma, B_1 \wedge B_2 \xrightarrow{K} A}\wedge\text{L}$

1. $\mathcal{E} :: \Gamma, B_1 \wedge B_2, A \xrightarrow{K} C$ (Assumption)

2. $\Gamma, B_1 \wedge B_2, B_1, B_2, A \xrightarrow{K} C$ (Weakening)

3. $\Gamma, B_1 \wedge B_2, B_1, B_2 \xrightarrow{K} C$ (i.h. on premise and 2)

4. $\Gamma, B_1 \wedge B_2 \xrightarrow{K} C$ (Rule ($\wedge$L))

Case. $\mathcal{D} = \dfrac{\Gamma, B_1 \vee B_2, B \xrightarrow{K} A \qquad \Gamma, B_1 \vee B_2, B_2 \xrightarrow{K} A}{\Gamma, B_1 \vee B_2 \xrightarrow{K} A}\vee\text{L}$

1. $\mathcal{E} :: \Gamma, B_1 \vee B_2, A \xrightarrow{K} C$ (Assumption)

2. $\Gamma, B_1 \vee B_2, B_1, A \xrightarrow{K} C$ (Weakening on 1)

3. $\Gamma, B_1 \vee B_2, B_1 \xrightarrow{K} C$ (i.h. on 1st premise and 2)

4. $\Gamma, B_1 \vee B_2, B_2, A \xrightarrow{K} C$ (Weakening on 1)

5. $\Gamma, B_1 \vee B_2, B_2 \xrightarrow{K} C$ (i.h. on 2nd premise and 4)

6. $\Gamma, B_1 \vee B_2 \xrightarrow{K} C$ (Rule ($\vee$L) on 3, 5)

Case. $\mathcal{D} = \dfrac{}{\Gamma, \bot \xrightarrow{K} A}\bot\mathrm{L}$

1. $\mathcal{E} :: \Gamma, \bot, A \xrightarrow{K} C$ (Assumption)

2. $\Gamma, \bot \xrightarrow{K} C$ (Rule ($\bot$L))

Case. $\mathcal{D} = \dfrac{\Gamma, B_1 \supset B_2 \xrightarrow{K} B_1 \qquad \Gamma, B_1 \supset B_2, B_2 \xrightarrow{K} A}{\Gamma, B_1 \supset B_2 \xrightarrow{K} A}\supset\mathrm{L}$

1. $\mathcal{E} :: \Gamma, B_1 \supset B_2, A \xrightarrow{K} C$ (Assumption)

2. $\Gamma, B_1 \supset B_2, B_2, A \xrightarrow{K} C$ (Weakening)

3. $\Gamma, B_1 \supset B_2, B_2 \xrightarrow{K} C$ (i.h. on 2nd premise and 2)

4. $\Gamma, B_1 \supset B_2 \xrightarrow{K} C$ (Rule ($\supset$L) on 1st premise and 3)

Cases of principal cuts. $\mathcal{D}$ ends in a right rule, and $\mathcal{E}$ ends in a left rule. Note that there are no principal cuts when $\mathcal{E}$ ends in (init) or ($\bot$L), because there are no right rules for atomic formulas or $\bot$. Similarly, there is no case for principal cut if $\mathcal{D}$ ends in ($\top$R) because $\top$ has no left rule. The case of principal cut when $\mathcal{E}$ ends in rule (claims) is covered in clause (2) of the theorem.

Case.

$$\mathcal{D} = \dfrac{\Gamma|_{K'} \xrightarrow{K'} A}{\Gamma \xrightarrow{K} K' \ \mathsf{says}\ A}\mathsf{saysR} \qquad\qquad \mathcal{E} = \dfrac{\Gamma, K' \ \mathsf{says}\ A, K' \ \mathsf{claims}\ A \xrightarrow{K} C}{\Gamma, K' \ \mathsf{says}\ A \xrightarrow{K} C}\mathsf{saysL}$$

1. $\Gamma, K' \ \mathsf{claims}\ A \xrightarrow{K} C$ (i.h. on $\mathcal{D}$ and premise of $\mathcal{E}$)

2. $\Gamma \xrightarrow{K} C$ (i.h.(2) on premise of $\mathcal{D}$ and 1)

Case.

$$\mathcal{D} = \dfrac{\Gamma \xrightarrow{K} A \qquad \Gamma \xrightarrow{K} B}{\Gamma \xrightarrow{K} A \wedge B}\wedge\mathrm{R} \qquad\qquad \mathcal{E} = \dfrac{\Gamma, A \wedge B, A, B \xrightarrow{K} C}{\Gamma, A \wedge B \xrightarrow{K} C}\wedge\mathrm{L}$$

1. $\Gamma, A, B \xrightarrow{K} C$ (i.h. on $\mathcal{D}$ and premise of $\mathcal{E}$)

2. $\Gamma, B \xrightarrow{K} C$ (i.h. on 1st premise of $\mathcal{D}$ and 1)

3. $\Gamma \xrightarrow{K} C$ (i.h. on 2nd premise of $\mathcal{D}$ and 2)

Case.

$$\mathcal{D} = \frac{\Gamma \xrightarrow{K} A}{\Gamma \xrightarrow{K} A \vee B} \vee \mathrm{R}_1 \qquad \mathcal{E} = \frac{\Gamma, A \vee B, A \xrightarrow{K} C \qquad \Gamma, A \vee B, B \xrightarrow{K} C}{\Gamma, A \vee B \xrightarrow{K} C} \vee \mathrm{L}$$

1. $\Gamma, A \xrightarrow{K} C$ (i.h. on $\mathcal{D}$ and 1st premise of $\mathcal{E}$)

2. $\Gamma \xrightarrow{K} C$ (i.h. on premise of $\mathcal{D}$ and 1)

Case.

$$\mathcal{D} = \frac{\Gamma \xrightarrow{K} B}{\Gamma \xrightarrow{K} A \vee B} \vee \mathrm{R}_2 \qquad \mathcal{E} = \frac{\Gamma, A \vee B, A \xrightarrow{K} C \qquad \Gamma, A \vee B, B \xrightarrow{K} C}{\Gamma, A \vee B \xrightarrow{K} C} \vee \mathrm{L}$$

1. $\Gamma, B \xrightarrow{K} C$ (i.h. on $\mathcal{D}$ and 2nd premise of $\mathcal{E}$)

2. $\Gamma \xrightarrow{K} C$ (i.h. on premise of $\mathcal{D}$ and 1)

Case.

$$\mathcal{D} = \frac{\Gamma, A \xrightarrow{K} B}{\Gamma \xrightarrow{K} A \supset B} \supset\mathrm{R} \qquad \mathcal{E} = \frac{\Gamma, A \supset B \xrightarrow{K} A \qquad \Gamma, A \supset B, B \xrightarrow{K} C}{\Gamma, A \supset B \xrightarrow{K} C} \supset\mathrm{L}$$

1. $\Gamma, B \xrightarrow{K} C$ (i.h. on $\mathcal{D}$ and 2nd premise of $\mathcal{E}$)

2. $\Gamma, A \xrightarrow{K} C$ (i.h. on premise of $\mathcal{D}$ and 1)

3. $\Gamma \xrightarrow{K} A$ (i.h. on $\mathcal{D}$ and 1st premise of $\mathcal{E}$)

4. $\Gamma \xrightarrow{K} C$ (i.h. on 3 and 2)

Proof of (2).

We analyze cases on the last rule of $\mathcal{E}$.

Case. $\mathcal{E} = \dfrac{P \text{ atomic}}{\Gamma, K \,\textsf{claims}\, A, P \xrightarrow{K} P}\text{init}$

1. $\Gamma, P \xrightarrow{K} P$ (Rule (init))

Case. $\mathcal{E} = \dfrac{\Gamma, K \text{ claims } A, A \xrightarrow{K'} C \qquad K \succeq K'}{\Gamma, K \text{ claims } A \xrightarrow{K'} C}\text{claims}$ (Principal cut)

1. $\mathcal{D} :: \Gamma|_K \xrightarrow{K} A$ (Assumption)
2. $\Gamma, A \xrightarrow{K'} C$ (i.h. on $\mathcal{D}$ and 1st premise of $\mathcal{E}$)
3. $\Gamma \xrightarrow{K} A$ (Weakening on 1)
4. $\Gamma \xrightarrow{K'} A$ (Subsumption (Theorem B.1) on 3 using $K \succeq K'$)
5. $\Gamma \xrightarrow{K'} C$ (i.h.(1) on 4 and 2)

Case. $\mathcal{E} = \dfrac{\Gamma, K \text{ claims } A, K'' \text{ claims } B, B \xrightarrow{K'} C \qquad K'' \succeq K'}{\Gamma, K \text{ claims } A, K'' \text{ claims } B \xrightarrow{K'} C}\text{claims}$ (Non-Principal cut)

1. $\Gamma, K'' \text{ claims } B, B \xrightarrow{K'} C$ (i.h. on $\mathcal{D}$ and 1st premise of $\mathcal{E}$)
2. $\Gamma, K'' \text{ claims } B \xrightarrow{K'} C$ (Rule (claims) on 1 and $K'' \succeq K'$)

Case. $\mathcal{E} = \dfrac{\Gamma|_{K''}, K \text{ claims } A \xrightarrow{K''} C}{\Gamma, K \text{ claims } A \xrightarrow{K'} K'' \text{ says } C}\text{saysR}$ $(K \succeq K'')$

1. $\mathcal{D} :: \Gamma|_K \xrightarrow{K} A$ (Assumption)
2. $\Gamma|_{K''}|_K = \Gamma|_K$ (Assumption $K \succeq K''$)
3. $\mathcal{D} :: \Gamma|_{K''}|_K \xrightarrow{K} A$ (From 1 and 2)
4. $\Gamma|_{K''} \xrightarrow{K''} C$ (i.h. on 3 and premise of $\mathcal{E}$)
5. $\Gamma \xrightarrow{K'} K'' \text{ says } C$ (Rule (saysR) on 4)

Case. $\mathcal{E} = \dfrac{\Gamma|_{K''} \xrightarrow{K''} C}{\Gamma, K \text{ claims } A \xrightarrow{K'} K'' \text{ says } C}\text{saysR}$ $(K \not\succeq K'')$

1. $\Gamma \xrightarrow{K'} K'' \text{ says } C$ (Rule (saysR) on premise of $\mathcal{E}$)

Case. $\mathcal{E} = \dfrac{\Gamma, K \text{ claims } A, K'' \text{ says } B, K'' \text{ claims } B \xrightarrow{K'} C}{\Gamma, K \text{ claims } A, K'' \text{ says } B \xrightarrow{K'} C}\text{saysL}$

1. $\mathcal{D} :: \Gamma|_K \xrightarrow{K} A$ (Assumption)

2. $(\Gamma, K'' \textsf{ says } B, K'' \textsf{ claims } B)|_K \xrightarrow{K} A$ (Possibly Weakening 1)

3. $\Gamma, K'' \textsf{ says } B, K'' \textsf{ claims } B \xrightarrow{K'} C$ (i.h. on 2 and premise of $\mathcal{E}$)

4. $\Gamma, K'' \textsf{ says } B \xrightarrow{K'} C$ (Rule (saysL))

Case. $\mathcal{E} = \dfrac{\Gamma, K \textsf{ claims } A \xrightarrow{K'} C_1 \qquad \Gamma, K \textsf{ claims } A \xrightarrow{K'} C_2}{\Gamma, K \textsf{ claims } A \xrightarrow{K'} C_1 \wedge C_2} \wedge\mathrm{R}$

1. $\Gamma \xrightarrow{K'} C_1$ (i.h. on $\mathcal{D}$ and 1st premise of $\mathcal{E}$)

2. $\Gamma \xrightarrow{K'} C_2$ (i.h. on $\mathcal{D}$ and 2nd premise of $\mathcal{E}$)

3. $\Gamma \xrightarrow{K'} C_1 \wedge C_2$ (Rule ($\wedge$R))

Case. $\mathcal{E} = \dfrac{\Gamma, K \textsf{ claims } A, B_1 \wedge B_2, B_1, B_2 \xrightarrow{K'} C}{\Gamma, K \textsf{ claims } A, B_1 \wedge B_2 \xrightarrow{K'} C} \wedge\mathrm{L}$

1. $\Gamma, B_1 \wedge B_2, B_1, B_2 \xrightarrow{K'} C$ (i.h. on $\mathcal{D}$ and premise of $\mathcal{E}$)

2. $\Gamma, B_1 \wedge B_2 \xrightarrow{K'} C$ (Rule ($\wedge$L))

Case. $\mathcal{E} = \dfrac{\Gamma, K \textsf{ claims } A \xrightarrow{K'} C_1}{\Gamma, K \textsf{ claims } A \xrightarrow{K'} C_1 \vee C_2} \vee \mathrm{R}_1$

1. $\Gamma \xrightarrow{K'} C_1$ (i.h. on $\mathcal{D}$ and premise of $\mathcal{E}$)

2. $\Gamma \xrightarrow{K'} C_1 \vee C_2$ (Rule ($\vee$ R_1))

Case. $\mathcal{E} = \dfrac{\Gamma, K \textsf{ claims } A \xrightarrow{K'} C_2}{\Gamma, K \textsf{ claims } A \xrightarrow{K'} C_1 \vee C_2} \vee \mathrm{R}_2$

1. $\Gamma \xrightarrow{K'} C_2$ (i.h. on $\mathcal{D}$ and premise of $\mathcal{E}$)

2. $\Gamma \xrightarrow{K'} C_1 \vee C_2$ (Rule ($\vee$ R_2))

Case. $\mathcal{E} = \dfrac{\Gamma, K \textsf{ claims } A, B_1 \vee B_2, B_1 \xrightarrow{K'} C \qquad \Gamma, K \textsf{ claims } A, B_1 \vee B_2, B_2 \xrightarrow{K'} C}{\Gamma, K \textsf{ claims } A, B_1 \vee B_2 \xrightarrow{K'} C} \vee \mathrm{L}$

1. $\Gamma, B_1 \vee B_2, B_1 \xrightarrow{K'} C$ (i.h. on $\mathcal{D}$ and 1st premise of $\mathcal{E}$)

2. $\Gamma, B_1 \vee B_2, B_2 \xrightarrow{K'} C$ (i.h. on $\mathcal{D}$ and 2nd premise of $\mathcal{E}$)

3. $\Gamma, B_1 \vee B_2 \xrightarrow{K'} C$ (Rule ($\vee$L))

Case. $\mathcal{E} = \dfrac{}{\Gamma, K \textsf{ claims } A \xrightarrow{K'} \top}\top\text{R}$

1. $\Gamma \xrightarrow{K'} \top$ (Rule ($\top$R))

Case. $\mathcal{E} = \dfrac{}{\Gamma, K \textsf{ claims } A, \bot \xrightarrow{K'} C}\bot\text{L}$

1. $\Gamma, \bot \xrightarrow{K'} C$ (Rule ($\bot$L))

Case. $\mathcal{E} = \dfrac{\Gamma, K \textsf{ claims } A, C_1 \xrightarrow{K'} C_2}{\Gamma, K \textsf{ claims } A \xrightarrow{K'} C_1 \supset C_2}\supset\text{R}$

1. $\Gamma, C_1 \xrightarrow{K'} C_2$ (i.h. on $\mathcal{D}$ and premise of $\mathcal{E}$)
2. $\Gamma \xrightarrow{K'} C_1 \supset C_2$ (Rule ($\supset$R))

Case. $\mathcal{E} = \dfrac{\Gamma, K \textsf{ claims } A, B_1 \supset B_2 \xrightarrow{K'} B_1 \qquad \Gamma, K \textsf{ claims } A, B_1 \supset B_2, B_2 \xrightarrow{K'} C}{\Gamma, K \textsf{ claims } A, B_1 \supset B_2 \xrightarrow{K'} C}\supset\text{L}$

1. $\Gamma, B_1 \supset B_2 \xrightarrow{K'} B_1$ (i.h. on $\mathcal{D}$ and 1st premise of $\mathcal{E}$)
2. $\Gamma, B_1 \supset B_2, B_2 \xrightarrow{K'} C$ (i.h. on $\mathcal{D}$ and 2nd premise of $\mathcal{E}$)
3. $\Gamma, B_1 \supset B_2 \xrightarrow{K'} C$ (Rule ($\supset$L))

$\square$

Theorem B.3 (Identity; Theorem 3.8)**.** *For each formula A, $\Gamma, A \xrightarrow{K} A$.*

Proof. By induction on A.

Case. $A = P$ (A is atomic)

1. $\Gamma, P \xrightarrow{K} P$ (Rule (init))

Case. $A = A_1 \wedge A_2$

1. $\Gamma, A_1 \wedge A_2, A_1, A_2 \xrightarrow{K} A_1$ (i.h.)
2. $\Gamma, A_1 \wedge A_2, A_1, A_2 \xrightarrow{K} A_2$ (i.h.)
3. $\Gamma, A_1 \wedge A_2, A_1, A_2 \xrightarrow{K} A_1 \wedge A_2$ (Rule ($\wedge$R) 1, 2)
4. $\Gamma, A_1 \wedge A_2 \xrightarrow{K} A_1 \wedge A_2$ (Rule ($\wedge$L) 3)

Case. $A = A_1 \vee A_2$

1. $\Gamma, A_1 \vee A_2, A_1 \xrightarrow{K} A_1$ (i.h.)
2. $\Gamma, A_1 \vee A_2, A_1 \xrightarrow{K} A_1 \vee A_2$ (Rule ($\vee$ R$_1$) 1)
3. $\Gamma, A_1 \vee A_2, A_2 \xrightarrow{K} A_2$ (i.h.)
4. $\Gamma, A_1 \vee A_2, A_2 \xrightarrow{K} A_1 \vee A_2$ (Rule ($\vee$ R$_2$) 3)
5. $\Gamma, A_1 \vee A_2 \xrightarrow{K} A_1 \vee A_2$ (Rule ($\vee$L) 2, 4)

Case. $A = \top$

1. $\Gamma, \top \xrightarrow{K} \top$ (Rule ($\top$R))

Case. $A = \bot$

1. $\Gamma, \bot \xrightarrow{K} \bot$ (Rule ($\bot$L))

Case. $A = A_1 \supset A_2$

1. $\Gamma, A_1 \supset A_2, A_1 \xrightarrow{K} A_1$ (i.h.)
2. $\Gamma, A_1 \supset A_2, A_1, A_2 \xrightarrow{K} A_2$ (i.h.)
3. $\Gamma, A_1 \supset A_2, A_1 \xrightarrow{K} A_2$ (Rule ($\supset$L))
4. $\Gamma, A_1 \supset A_2 \xrightarrow{K} A_1 \supset A_2$ (Rule ($\supset$R))

Case. $A = K' \text{ says } B$

1. $\Gamma|_{K'}, K' \text{ claims } B, B \xrightarrow{K'} B$ (i.h.)
2. $\Gamma|_{K'}, K' \text{ claims } B \xrightarrow{K'} B$ (Rule (claims))
3. $\Gamma, K' \text{ says } B, K' \text{ claims } B \xrightarrow{K} K' \text{ says } B$ (Rule (saysR))
4. $\Gamma, K' \text{ says } B \xrightarrow{K} K' \text{ says } B$ (Rule (saysL))

□

C Proof of Equivalence from Section 3.6

The objective of this section is to prove Theorem 3.9, showing that the axiomatic, natural deduction, and sequent calculus proof systems are equivalent. Although it is possible to show that natural deduction and sequent calculus are equivalent without reference to the axiomatic system, we do not do this here, and prove the equivalence of the three systems simultaneously. First, we expand the theory of the axiomatic system.

C.1 The Axiomatic System for DTL_0

In Section 2, we presented some rules and axioms for the axiomatic system. Here, we list all the rules and axioms, including those listed earlier.

$$\frac{\vdash_H A}{\vdash_H K \mathsf{\ says\ } A}\text{nec} \qquad \frac{\vdash_H A \supset B \qquad \vdash_H A}{\vdash_H B}\text{mp} \qquad \frac{A \text{ is an axiom}}{\vdash_H A}\text{ax}$$

Axioms:

$(K \mathsf{\ says\ } (A \supset B)) \supset ((K \mathsf{\ says\ } A) \supset (K \mathsf{\ says\ } B))$	(K)
$(K \mathsf{\ says\ } A) \supset K \mathsf{\ says\ } K \mathsf{\ says\ } A$	(4)
$K \mathsf{\ says\ } ((K \mathsf{\ says\ } A) \supset A)$	(C)
$(K_1 \mathsf{\ says\ } A) \supset (K_2 \mathsf{\ says\ } A)$ if $K_1 \succeq K_2$	(S)
$A \supset (B \supset A)$	(imp1)
$(A \supset B) \supset ((A \supset (B \supset C)) \supset (A \supset C))$	(imp2)
$A \supset (B \supset (A \wedge B))$	(conj1)
$(A \wedge B) \supset A$	(conj2)
$(A \wedge B) \supset B$	(conj3)
$A \supset (A \vee B)$	(disj1)
$B \supset (A \vee B)$	(disj2)
$(A \supset C) \supset ((B \supset C) \supset ((A \vee B) \supset C))$	(disj3)
$\top$	(true)
$\bot \supset A$	(false)

Next, we introduce a *generalized* axiomatic system, to reason from hypothesis. Let Γ denote a multi set of formulas (not judgments). We write $\Gamma \vdash_G A$ to mean that A may be established from assumptions Γ. The rules of the generalized axiomatic system are:

$$\frac{}{\Gamma, A \vdash_G A}\text{use} \qquad \frac{\cdot \vdash_G A}{\Gamma \vdash_G K \mathsf{\ says\ } A}\text{nec} \qquad \frac{\Gamma \vdash_G A \supset B \qquad \Gamma \vdash_G A}{\Gamma \vdash_G B}\text{mp}$$

$$\frac{A \text{ is an axiom}}{\Gamma \vdash_G A}\text{ax}$$

Now we prove some basic properties of the generalized axiomatic system, including the deduction theorem, and show that the generalized system reduces to the axiomatic system when Γ is empty.

Lemma C.1 (Basic properties)**.** *The following hold.*

1. *(Weakening)* $\Gamma \vdash_G A$ *implies* $\Gamma, \Gamma' \vdash_G A$
2. *(Substitution)* $\Gamma \vdash_G A$ *and* $\Gamma, A \vdash_G B$ *imply* $\Gamma \vdash_G B$

Proof. (1) follows by an easy induction on the derivation of $\Gamma \vdash_G A$ (details omitted here). (2) follows by induction on the derivation of $\Gamma, A \vdash_G B$. We analyze the last rule in the derivation.

Case. $\dfrac{}{\Gamma, A \vdash_G A}\text{use}$ (Principal case)

1. $\Gamma \vdash_G A$ (First given derivation)

Case. $\dfrac{}{\Gamma, A, B \vdash_G B}\text{use}$ (Non-principal case)

1. $\Gamma, B \vdash_G B$ (Rule (use))

Case. $\dfrac{\cdot \vdash_G B}{\Gamma, A \vdash_G K \text{ says } B}\text{nec}$

1. $\cdot \vdash_G B$ (Premise)
2. $\Gamma \vdash_G K \text{ says } B$ (Rule (nec))

Case. $\dfrac{\Gamma, A \vdash_G B_1 \supset B_2 \qquad \Gamma, A \vdash_G B_1}{\Gamma, A \vdash_G B_2}\text{mp}$

1. $\Gamma \vdash_G B_1 \supset B_2$ (i.h. on 1st premise)
2. $\Gamma \vdash_G B_1$ (i.h. on 2nd premise)
3. $\Gamma \vdash_G B_2$ (Rule (mp))

Case. $\dfrac{B \text{ is an axiom}}{\Gamma, A \vdash_G B}\text{ax}$

1. $\Gamma \vdash_G B$ (Rule (ax))

□

Theorem C.2 (Deduction)**.** *The following hold.*

1. $\Gamma \vdash_G A \supset B$ *implies* $\Gamma, A \vdash_G B$
2. $\Gamma, A \vdash_G B$ *implies* $\Gamma \vdash_G A \supset B$

Proof. We prove (1) first. Assume $\Gamma \vdash_G A \supset B$. Then we have

1. $\Gamma, A \vdash_G A \supset B$ (Weakening)
2. $\Gamma, A \vdash_G A$ (Rule (use))
3. $\Gamma, A \vdash_G B$ (Rule (mp))

Next we prove (2). We induct on the derivation of $\Gamma, A \vdash_G B$, case analyzing the last rule.

Case. $\dfrac{}{\Gamma, A \vdash_G A}\text{use}$ (Principal case)

Here $A = B$, and we must show that $\Gamma \vdash_G A \supset A$.

1. $\Gamma \vdash_G (A \supset (B \supset A)) \supset ((A \supset ((B \supset A) \supset A)) \supset (A \supset A))$ (Rule (ax) and imp2)

2. $\Gamma \vdash_G A \supset (B \supset A)$ (Rule (ax) and imp1)

3. $\Gamma \vdash_G (A \supset ((B \supset A) \supset A)) \supset (A \supset A)$ (Rule (mp))

4. $\Gamma \vdash_G A \supset ((B \supset A) \supset A)$ (Rule (ax) and imp1)

5. $\Gamma \vdash_G A \supset A$ (Rule (mp) on 3 and 4)

Case. $\dfrac{}{\Gamma, A, B \vdash_G B}\text{use}$ (Non-principal case)

1. $\Gamma, B \vdash_G B \supset (A \supset B)$ (Rule (ax) and imp1)
2. $\Gamma, B \vdash_G B$ (Rule (use))
3. $\Gamma, B \vdash_G A \supset B$ (Rule (mp))

Case. $\dfrac{\cdot \vdash_G B}{\Gamma, A \vdash_G K \textsf{ says } B}\text{nec}$

1. $\Gamma \vdash_G K \textsf{ says } B$ (Rule (nec) on premise)
2. $\Gamma \vdash_G (K \textsf{ says } B) \supset (A \supset K \textsf{ says } B)$ (Rule (ax) and imp1)
3. $\Gamma \vdash_G A \supset K \textsf{ says } B$ (Rule mp)

Case. $\dfrac{\Gamma, A \vdash_G B_1 \supset B_2 \qquad \Gamma, A \vdash_G B_1}{\Gamma, A \vdash_G B_2}\text{mp}$

1. $\Gamma \vdash_G A \supset (B_1 \supset B_2)$ (i.h. on premise 1)
2. $\Gamma \vdash_G A \supset B_1$ (i.h. on premise 2)
3. $\Gamma \vdash_G (A \supset B_1) \supset ((A \supset (B_1 \supset B_2)) \supset (A \supset B_2))$ (Rule (ax) and imp2)
4. $\Gamma \vdash_G (A \supset (B_1 \supset B_2)) \supset (A \supset B_2)$ (Rule (mp) on 3 and 2)
5. $\Gamma \vdash_G A \supset B_2$ (Rule (mp) on 4 and 1)

Case. $\dfrac{B \text{ is an axiom}}{\Gamma, A \vdash_G B}\text{ax}$

1. $\Gamma \vdash_G B \supset (A \supset B)$ (Rule (ax) and imp1)
2. $\Gamma \vdash_G B$ (Rule (ax))
3. $\Gamma \vdash_G A \supset B$ (Rule (mp))

□

Theorem C.3 (G iff H). $\vdash_H A$ *if and only if* $\cdot \vdash_G A$

Proof. In each direction by straightforward induction on the given derivation. □

Lemma C.4 (Currying). *$\Gamma \vdash_G (A \wedge B) \supset C$ if and only if $\Gamma \vdash_G A \supset (B \supset C)$.*

Proof. ("If" direction)

1. $\Gamma \vdash_G A \supset (B \supset C)$ (Assumption)
2. $\Gamma, A, B \vdash_G C$ (Theorem C.2 twice)
3. $\Gamma, A \wedge B \vdash_G (A \wedge B) \supset A$ (Rule (ax) and conj1)
4. $\Gamma, A \wedge B \vdash_G (A \wedge B)$ (Rule (use))
5. $\Gamma, A \wedge B \vdash_G A$ (Rule (mp) on 3 and 4)
6. $\Gamma, A \wedge B \vdash_G (A \wedge B) \supset B$ (Rule (ax) and conj2)
7. $\Gamma, A \wedge B \vdash_G B$ (Rule (mp) on 6 and 4)
8. $\Gamma, A \wedge B, B \vdash_G C$ (Substitution Lemma C.1 on 5 and 2)
9. $\Gamma, A \wedge B \vdash_G C$ (Substitution Lemma C.1 on 7 and 8)
10. $\Gamma \vdash_G (A \wedge B) \supset C$ (Theorem C.2)

("Only if" direction)

1. $\Gamma \vdash_G (A \wedge B) \supset C$ (Assumption)
2. $\Gamma, A \wedge B \vdash_G C$ (Theorem C.2)
3. $\Gamma, A, B \vdash_G A \supset (B \supset (A \wedge B))$ (Rule (ax) and conj3)
4. $\Gamma, A, B \vdash_G A$ (Rule (use))
5. $\Gamma, A, B \vdash_G B \supset (A \wedge B)$ (Rule (mp) on 3 and 4)
6. $\Gamma, A, B \vdash_G B$ (Rule (use))
7. $\Gamma, A, B \vdash_G (A \wedge B)$ (Rule (mp) on 5 and 6)
8. $\Gamma, A, B \vdash_G C$ (Substitution Lemma C.1 on 7 and 2)
9. $\Gamma, A \vdash_G B \supset C$ (Theorem C.2)
10. $\Gamma \vdash_G A \supset (B \supset C)$ (Theorem C.2)

□

Lemma C.5. *$K' \succeq K$ and $\cdot \vdash_G K'$ says A imply $\cdot \vdash_G K$ says K' says A*

Proof.

1. $\cdot \vdash_G (K' \mathsf{\ says\ } A) \supset K' \mathsf{\ says\ } K' \mathsf{\ says\ } A$ (Rule (ax) and Axiom 4)
2. $K' \mathsf{\ says\ } A \vdash_G K' \mathsf{\ says\ } K' \mathsf{\ says\ } A$ (Theorem C.2)
3. $K' \mathsf{\ says\ } A \vdash_G (K' \mathsf{\ says\ } K' \mathsf{\ says\ } A) \supset (K \mathsf{\ says\ } K' \mathsf{\ says\ } A)$ (Rule (ax) and S)
4. $K' \mathsf{\ says\ } A \vdash_G K \mathsf{\ says\ } K' \mathsf{\ says\ } A$ (Rule (mp) on 3 and 2)
5. $\cdot \vdash_G (K' \mathsf{\ says\ } A) \supset K \mathsf{\ says\ } K' \mathsf{\ says\ } A$ (Theorem C.2)

□

C.2 Proof of Equivalence

Let $\overline{\Gamma}$ denote the reification of the Γ as a formula:

$$\begin{array}{rcl} \overline{\cdot} & = & \top \\ \overline{\Gamma, A \text{ true}} & = & \overline{\Gamma} \wedge A \\ \overline{\Gamma, K \text{ claims } A} & = & \overline{\Gamma} \wedge (K \text{ says } A) \end{array}$$

Lemma C.6 (Natural Deduction $\Rightarrow$ Axiomatic)**.** $\Gamma \vdash^K A$ *implies* $\cdot \vdash_G K \text{ says } (\overline{\Gamma} \supset A)$

Proof. We induct on the derivation of $\Gamma \vdash^K A$, analyzing cases on the last rule. Some of the cases related to says and claims are shown below. Others are straightforward. To keep proofs short, we freely use properties such as Currying (Lemma C.4) and $(K \text{ says } (A \wedge B)) \equiv ((K \text{ says } A) \wedge (K \text{ says } B))$, without explicit mention.

Case. $\dfrac{}{\Gamma, A \vdash^K A}\text{hyp}$

1. $\overline{\Gamma} \vdash_G A \supset A$ (See proof of Theorem C.2.2; case (use))
2. $\cdot \vdash_G \overline{\Gamma} \supset (A \supset A)$ (Theorem C.2)
3. $\cdot \vdash_G (\overline{\Gamma} \wedge A) \supset A$ (Lemma C.4)
4. $\cdot \vdash_G K \text{ says } ((\overline{\Gamma} \wedge A) \supset A)$ (Rule (nec))

Case. $\dfrac{K' \succeq K}{\Gamma, K' \text{ claims } A \vdash^K A}\text{claims}$

1. $\cdot \vdash_G K' \text{ says } ((K' \text{ says } A) \supset A)$ (Rule (ax) and C)
2. $\cdot \vdash_G ((K' \text{ says } A) \supset A) \supset ((\overline{\Gamma} \wedge (K' \text{ says } A)) \supset A)$ (Theorem in G)
3. $\cdot \vdash_G K' \text{ says } (((K' \text{ says } A) \supset A) \supset ((\overline{\Gamma} \wedge (K' \text{ says } A)) \supset A))$ (Rule (nec) on 2)
4. $\cdot \vdash_G (K' \text{ says } ((K' \text{ says } A) \supset A)) \supset K' \text{ says } ((\overline{\Gamma} \wedge (K' \text{ says } A)) \supset A)$
   (Rule (ax), K and (mp) on 3)
5. $\cdot \vdash_G K' \text{ says } ((\overline{\Gamma} \wedge (K' \text{ says } A)) \supset A)$ (Rule (mp) on 4 and 1)
6. $\cdot \vdash_G (K' \text{ says } ((\overline{\Gamma} \wedge (K' \text{ says } A)) \supset A)) \supset K \text{ says } ((\overline{\Gamma} \wedge (K' \text{ says } A)) \supset A)$
   (Rule (ax) and S; $K' \succeq K$)
7. $\cdot \vdash_G K \text{ says } ((\overline{\Gamma} \wedge (K' \text{ says } A)) \supset A)$ (Rule (mp) on 6 and 5)

Case. $\dfrac{\Gamma|_{K'} \vdash^{K'} A}{\Gamma \vdash^K K' \text{ says } A}\text{saysI}$

Let $\Gamma|_{K'} = K_1 \text{ claims } A_1, \ldots, K_n \text{ claims } A_n$. Then $K_i \succeq K'$ $(1 \leq i \leq n)$

1. $\cdot \vdash_G K' \text{ says } (((K_1 \text{ says } A_1) \wedge \ldots \wedge (K_n \text{ says } A_n)) \supset A)$ (i.h. on premise)

2. $\cdot \vdash_G ((K' \textsf{ says } K_1 \textsf{ says } A_1) \wedge \ldots \wedge (K' \textsf{ says } K_n \textsf{ says } A_n)) \supset K' \textsf{ says } A$

(Rule (ax), K and (mp))

3. $K' \textsf{ says } K_1 \textsf{ says } A_1, \ldots, K' \textsf{ says } K_n \textsf{ says } A_n \vdash_G K' \textsf{ says } A$ (Theorem C.2)

4. $\cdot \vdash_G (K_i \textsf{ says } A_i) \supset K' \textsf{ says } K_i \textsf{ says } A_i$ (Lemma C.5)

5. $K_i \textsf{ says } A_i \vdash_G K' \textsf{ says } K_i \textsf{ says } A_i$ (Theorem C.2)

6. $K_1 \textsf{ says } A_1, \ldots, K_n \textsf{ says } A_n \vdash_G K' \textsf{ says } A$ (Substitution Lemma C.1 on 5 and 3)

7. $\overline{\Gamma} \vdash_G K' \textsf{ says } A$ (Weakening Lemma C.1)

8. $\cdot \vdash_G \overline{\Gamma} \supset K' \textsf{ says } A$ (Theorem C.2)

9. $\cdot \vdash_G K \textsf{ says } (\overline{\Gamma} \supset K' \textsf{ says } A)$ (Rule (nec))

Case. $\dfrac{\Gamma \vdash^K K' \textsf{ says } B \qquad \Gamma, K' \textsf{ claims } B \vdash^K A}{\Gamma \vdash^K A}\textsf{saysE}$

1. $\cdot \vdash_G K \textsf{ says } (\overline{\Gamma} \supset K' \textsf{ says } B)$ (i.h. on 1st premise)

2. $\cdot \vdash_G K \textsf{ says } (\overline{\Gamma} \supset ((K' \textsf{ says } B) \supset A))$ (i.h. on 2nd premise)

3. $\cdot \vdash_G (\overline{\Gamma} \supset K' \textsf{ says } B) \supset ((\overline{\Gamma} \supset ((K' \textsf{ says } B) \supset A)) \supset (\overline{\Gamma} \supset A))$ (Rule (ax) and imp2)

4. $\cdot \vdash_G K \textsf{ says } ((\overline{\Gamma} \supset K' \textsf{ says } B) \supset ((\overline{\Gamma} \supset ((K' \textsf{ says } B) \supset A)) \supset (\overline{\Gamma} \supset A)))$ (Rule (nec))

5. $\cdot \vdash_G (K \textsf{ says } (\overline{\Gamma} \supset K' \textsf{ says } B)) \supset ((K \textsf{ says } (\overline{\Gamma} \supset ((K' \textsf{ says } B) \supset A))) \supset K \textsf{ says } (\overline{\Gamma} \supset A))$ (Rule (ax), K and (mp))

6. $\cdot \vdash_G (K \textsf{ says } (\overline{\Gamma} \supset ((K' \textsf{ says } B) \supset A))) \supset K \textsf{ says } (\overline{\Gamma} \supset A)$ (Rule (mp) on 5 and 1)

7. $\cdot \vdash_G K \textsf{ says } (\overline{\Gamma} \supset A)$ (Rule (mp) on 6 and 2)

□

Lemma C.7 (Axiomatic ⇒ Sequent Calculus). $\vdash_H A$ *implies* $\cdot \xrightarrow{K} A$ *for each* K*.*

Proof. We induct on the derivation of $\vdash_H A$, and analyze cases on the last rule in the derivation.

Case. $\dfrac{\vdash_H A}{\vdash_H K' \textsf{ says } A}\text{nec}$

1. $\cdot \xrightarrow{K'} A$ (i.h. on premise with K')

2. $\cdot \xrightarrow{K} K' \textsf{ says } A$ (Rule (saysR))

Case. $\dfrac{\vdash_H A \supset B \qquad \vdash_H A}{\vdash_H B}\text{mp}$

1. $\cdot \xrightarrow{K} A \supset B$ (i.h. on 1st premise)
2. $\cdot \xrightarrow{K} A$ (i.h. on 2nd premise)
3. $A \supset B, A \xrightarrow{K} A$ (Theorem B.3)
4. $A \supset B, A, B \xrightarrow{K} B$ (Theorem B.3)
5. $A \supset B, A \xrightarrow{K} B$ (Rule ($\supset$L) on 3 and 4)
6. $A \xrightarrow{K} B$ (Theorem B.2 on 1 and 5)
7. $\cdot \xrightarrow{K} B$ (Theorem B.2 on 2 and 6)

Case. $\dfrac{A \text{ is an axiom}}{\vdash_H H}\text{ax}$

We case analyze all axioms A, in each case showing that $\cdot \xrightarrow{K} A$. Some representative cases are shown below (others are straightforward, since they use only the laws of propositional logic)

Case. (Axiom K) $A = (K' \mathsf{~says~} (A' \supset B')) \supset ((K' \mathsf{~says~} A') \supset (K' \mathsf{~says~} B'))$

1. $K' \mathsf{~claims~} (A' \supset B'), K' \mathsf{~claims~} A', A' \supset B', A' \xrightarrow{K'} A'$ (Theorem B.3)
2. $K' \mathsf{~claims~} (A' \supset B'), K' \mathsf{~claims~} A', A' \supset B', A', B' \xrightarrow{K'} B'$ (Theorem B.3)
3. $K' \mathsf{~claims~} (A' \supset B'), K' \mathsf{~claims~} A', A' \supset B', A' \xrightarrow{K'} B'$ (Rule ($\supset$L))
4. $K' \mathsf{~claims~} (A' \supset B'), K' \mathsf{~claims~} A' \xrightarrow{K'} B'$ (Rule (claims) twice)
5. $K' \mathsf{~says~} (A' \supset B'), K' \mathsf{~says~} A', K' \mathsf{~claims~} (A' \supset B'), K' \mathsf{~claims~} A' \xrightarrow{K} K' \mathsf{~says~} B'$

   (Rule (saysR))
6. $K' \mathsf{~says~} (A' \supset B'), K' \mathsf{~says~} A' \xrightarrow{K} K' \mathsf{~says~} B'$ (Rule (saysL) twice)
7. $\cdot \xrightarrow{K} (K' \mathsf{~says~} (A' \supset B')) \supset ((K' \mathsf{~says~} A') \supset (K' \mathsf{~says~} B'))$ (Rule ($\supset$R) twice)

Case. (Axiom 4) $A = (K' \mathsf{~says~} A') \supset K' \mathsf{~says~} K' \mathsf{~says~} A'$

1. $K' \mathsf{~claims~} A', A' \xrightarrow{K'} A'$ (Theorem B.3)
2. $K' \mathsf{~claims~} A' \xrightarrow{K'} A'$ (Rule (claims))
3. $K' \mathsf{~claims~} A' \xrightarrow{K'} K' \mathsf{~says~} A$ (Rule (saysR))
4. $K' \mathsf{~says~} A', K' \mathsf{~claims~} A' \xrightarrow{K} K' \mathsf{~says~} K' \mathsf{~says~} A$ (Rule (saysR))
5. $K' \mathsf{~says~} A' \xrightarrow{K} K' \mathsf{~says~} K' \mathsf{~says~} A$ (Rule (saysL))

6. $\cdot \xrightarrow{K} (K' \text{ says } A') \supset K' \text{ says } K' \text{ says } A'$ (Rule ($\supset$R))

Case. (Axiom C) $A = K' \text{ says } ((K' \text{ says } A') \supset A')$

1. $K' \text{ says } A', K' \text{ claims } A', A' \xrightarrow{K'} A'$ (Theorem B.3)
2. $K' \text{ says } A', K' \text{ claims } A' \xrightarrow{K'} A'$ (Rule (claims))
3. $K' \text{ says } A' \xrightarrow{K'} A'$ (Rule (saysL))
4. $\cdot \xrightarrow{K'} (K' \text{ says } A') \supset A'$ (Rule ($\supset$R))
5. $\cdot \xrightarrow{K} K' \text{ says } ((K' \text{ says } A') \supset A')$ (Rule (saysR))

Case. (Axiom S) $A = (K_1 \text{ says } A') \supset (K_2 \text{ says } A')$ and $K_1 \succeq K_2$

1. $K_1 \text{ claims } A', A' \xrightarrow{K_2} A'$ (Theorem B.3)
2. $K_1 \text{ claims } A' \xrightarrow{K_2} A'$ (Rule (claims); $K_1 \succeq K_2$)
3. $K_1 \text{ claims } A', K_1 \text{ says } A' \xrightarrow{K} K_2 \text{ says } A'$ (Rule (saysR); $K_1 \succeq K_2$)
4. $K_1 \text{ says } A' \xrightarrow{K} K_2 \text{ says } A'$ (Rule (saysL))
5. $\cdot \xrightarrow{K} (K_1 \text{ says } A') \supset (K_2 \text{ says } A')$ (Rule ($\supset$R))

□

Lemma C.8 (Sequent Calculus $\Rightarrow$ Natural Deduction)**.** *$\Gamma \xrightarrow{K} A$ implies $\Gamma \vdash^K A$.*

Proof. We induct on the derivation of $\Gamma \xrightarrow{K} A$ and analyze the last rule in the derivation.

Case. $$\frac{P \text{ atomic}}{\Gamma, P \xrightarrow{K} P}\text{init}$$

1. $\Gamma, P \vdash^K P$ (Rule (hyp))

Case. $$\frac{\Gamma, K \text{ claims } A, A \xrightarrow{K'} C \qquad K \succeq K'}{\Gamma, K \text{ claims } A \xrightarrow{K'} C}\text{claims}$$

1. $\Gamma, K \text{ claims } A, A \vdash^{K'} C$ (i.h. on premise)
2. $\Gamma, K \text{ claims } A \vdash^{K'} A$ (Rule (claims); $K \succeq K'$)
3. $\Gamma, K \text{ claims } A \vdash^{K'} C$ (Substitution Theorem 3.2 on 2 and 1)

Case. $$\frac{\Gamma|_K \xrightarrow{K} A}{\Gamma \xrightarrow{K'} K \text{ says } A}\text{saysR}$$

1. $\Gamma|_K \vdash^K A$ (i.h. on premise)

2. $\Gamma \vdash^{K'} K \mathsf{~says~} A$ (Rule (saysI))

Case. $$\frac{\Gamma, K \mathsf{~says~} A, K \mathsf{~claims~} A \xrightarrow{K'} C}{\Gamma, K \mathsf{~says~} A \xrightarrow{K'} C}\mathsf{saysL}$$

1. $\Gamma, K \mathsf{~says~} A, K \mathsf{~claims~} A \vdash^{K'} C$ (i.h. on premise)
2. $\Gamma, K \mathsf{~says~} A \vdash^{K'} K \mathsf{~says~} A$ (Rule (hyp))
3. $\Gamma, K \mathsf{~says~} A \vdash^{K'} C$ (Rule (saysE) on 2 and 1)

Case. $$\frac{\Gamma \xrightarrow{K} A \qquad \Gamma \xrightarrow{K} B}{\Gamma \xrightarrow{K} A \wedge B}\wedge\mathrm{R}$$

1. $\Gamma \vdash^{K} A$ (i.h. on 1st premise)
2. $\Gamma \vdash^{K} B$ (i.h. on 2nd premise)
3. $\Gamma \vdash^{K} A \wedge B$ (Rule ($\wedge$R))

Case. $$\frac{\Gamma, A \wedge B, A, B \xrightarrow{K} C}{\Gamma, A \wedge B \xrightarrow{K} C}\wedge\mathrm{L}$$

1. $\Gamma, A \wedge B, A, B \vdash^{K} C$ (i.h. on premise)
2. $\Gamma, A \wedge B \vdash^{K} A \wedge B$ (Rule (hyp))
3. $\Gamma, A \wedge B \vdash^{K} A$ (Rule ($\wedge$ E$_1$) on 2)
4. $\Gamma, A \wedge B \vdash^{K} B$ (Rule ($\wedge$ E$_2$) on 2)
5. $\Gamma, A \wedge B, B \vdash^{K} C$ (Substitution Theorem 3.2 on 3 and 1)
6. $\Gamma, A \wedge B \vdash^{K} C$ (Substitution Theorem 3.2 on 5 and 4)

Case. $$\frac{\Gamma \xrightarrow{K} A}{\Gamma \xrightarrow{K} A \vee B}\vee\mathrm{R}_1$$

1. $\Gamma \vdash^{K} A$ (i.h. on premise)
2. $\Gamma \vdash^{K} A \vee B$ (Rule $\vee$ I$_1$)

Case. $$\frac{\Gamma \xrightarrow{K} B}{\Gamma \xrightarrow{K} A \vee B}\vee\mathrm{R}_2$$

1. $\Gamma \vdash^{K} B$ (i.h. on premise)
2. $\Gamma \vdash^{K} A \vee B$ (Rule $\vee$ I$_2$)

Case. $$\dfrac{\Gamma, A \vee B, A \xrightarrow{K} C \qquad \Gamma, A \vee B, B \xrightarrow{K} C}{\Gamma, A \vee B \xrightarrow{K} C} \vee \mathrm{L}$$

1. $\Gamma, A \vee B, A \vdash^K C$ (i.h. on 1st premise)
2. $\Gamma, A \vee B, B \vdash^K C$ (i.h. on 2nd premise)
3. $\Gamma, A \vee B \vdash^K A \vee B$ (Rule (hyp))
4. $\Gamma, A \vee B \vdash^K C$ (Rule ($\vee$E) on 3, 1 and 2)

Case. $$\dfrac{}{\Gamma \xrightarrow{K} \top} \top\mathrm{R}$$

1. $\Gamma \vdash^K \top$ (Rule ($\top$I))

Case. $$\dfrac{}{\Gamma, \bot \xrightarrow{K} C} \bot\mathrm{L}$$

1. $\Gamma, \bot \vdash^K \bot$ (Rule (hyp))
2. $\Gamma, \bot \vdash^K C$ (Rule ($\bot$E))

Case. $$\dfrac{\Gamma, A \xrightarrow{K} B}{\Gamma \xrightarrow{K} A \supset B} \supset\mathrm{R}$$

1. $\Gamma, A \vdash^K B$ (i.h. on premise)
2. $\Gamma \vdash^K A \supset B$ (Rule ($\supset$I))

Case. $$\dfrac{\Gamma, A \supset B \xrightarrow{K} A \qquad \Gamma, A \supset B, B \xrightarrow{K} C}{\Gamma, A \supset B \xrightarrow{K} C} \supset\mathrm{L}$$

1. $\Gamma, A \supset B \vdash^K A$ (i.h. on 1st premise)
2. $\Gamma, A \supset B, B \vdash^K C$ (i.h. on 2nd premise)
3. $\Gamma, A \supset B \vdash^K A \supset B$ (Rule (hyp))
4. $\Gamma, A \supset B \vdash^K B$ (Rule ($\supset$E) on 3 and 1)
5. $\Gamma, A \supset B \vdash^K C$ (Substitution Theorem 3.2 on 4 and 2)

$\square$

Lemma C.9 (Equivalence)**.** *The following are equivalent for any Γ, K, and A.*

1. *$\Gamma \vdash^K A$ in the natural deduction system.*
2. *$\Gamma \xrightarrow{K} A$ in the sequent calculus.*

3. $\vdash_H K$ says $(\overline{\Gamma} \supset A)$ *in the axiomatic system.*

Proof. We show that (2) => (1) => (3) => (2).

Proof of (2) => (1). Follows immediately from Lemma C.8.

Proof of (1) => (3). Suppose $\Gamma \vdash^K A$. By Lemma C.6, $\cdot \vdash_G K$ says $(\overline{\Gamma} \supset A)$. By Theorem C.3, $\vdash_H K$ says $(\overline{\Gamma} \supset A)$.

Proof of (3) => (2). Suppose $\vdash_H K$ says $(\overline{\Gamma} \supset A)$. By Lemma C.7, $\cdot \xrightarrow{K} K$ says $(\overline{\Gamma} \supset A)$. Now observe that K says $(\overline{\Gamma} \supset A), \Gamma \xrightarrow{K} A$. Hence, by cut (Theorem B.2) we get $\Gamma \xrightarrow{K} A$.

□

Corollary C.10 (Equivalence; Theorem 3.9)**.** *The following are equivalent for any K, and A.*

1. $\cdot \vdash^K A$ *in the natural deduction system.*
2. $\cdot \xrightarrow{K} A$ *in the sequent calculus.*
3. $\vdash_H K$ says A *in the axiomatic system.*

Proof. Choose $\Gamma = \cdot$ in Lemma C.9. We get $\cdot \vdash^K A$ iff $\cdot \xrightarrow{K} A$ iff $\vdash_H K$ says $(\top \supset A)$. It only remains to show that $\vdash_H K$ says A if and only if $\vdash_H K$ says $(\top \supset A)$.

("If" direction)

1. $\vdash_H K$ says $(\top \supset A)$ (Assumption)
2. $\vdash_H \top \supset ((\top \supset A) \supset \top)$ (Rule (ax) and imp1)
3. $\vdash_H \top$ (Rule (ax) and true)
4. $\vdash_H (\top \supset A) \supset \top$ (Rule (mp) on 2 and 3)
5. $\vdash_H (\top \supset A) \supset (\top \supset A)$ (see proof of Theorem C.2.2; case (use))
6. $\vdash_H ((\top \supset A) \supset \top) \supset (((\top \supset A) \supset (\top \supset A)) \supset ((\top \supset A) \supset A))$ (Rule (ax) and imp2)
7. $\vdash_H ((\top \supset A) \supset (\top \supset A)) \supset ((\top \supset A) \supset A)$ (Rule (mp) on 6 and 4)
8. $\vdash_H ((\top \supset A) \supset A)$ (Rule (mp) on 7 and 5)
9. $\vdash_H K$ says $((\top \supset A) \supset A)$ (Rule (nec))
10. $\vdash_H (K$ says $(\top \supset A)) \supset K$ says A (Rule (ax), K and (mp) on 9)
11. $\vdash_H K$ says A (Rule (mp) on 10 and 1)

("Only if" direction)

1. $\vdash_H K \text{ says } A$ (Assumption)

2. $\vdash_H A \supset (\top \supset A)$ (Rule (ax) and imp1)

3. $\vdash_H K \text{ says } (A \supset (\top \supset A))$ (Rule (nec))

4. $\vdash_H (K \text{ says } A) \supset K \text{ says } (\top \supset A)$ (Rule (ax), K and (mp) on 3)

5. $\vdash_H K \text{ says } (\top \supset A)$ (Rule (mp) on 4 and 1)

$\square$

D Proofs from Section 4

D.1 Soundness

We show here that the Kripke semantics for DTL_0 are sound, i.e, if $\Gamma \xrightarrow{K} A$, then $M \models^K (\overline{\Gamma} \supset A)$ in each model M, where $\overline{\Gamma}$ is the context Γ reified as a formula, as defined in Appendix C.2. First, we prove a few preliminary lemmas. As a convention, we write w^K to denote a world w that is visible to K.

Lemma D.1 (Monotonicity). *If $w \models A$ and $w' \geq w$, then $w' \models A$.*

Proof. We induct on A.

Case. $A = P$ (A is atomic)
Suppose $w \models P$, and $w' \geq w$. We want to show that $w' \models P$. By assumption $w \models P$, which implies that $P \in \rho(w)$. Hence $P \in \rho(w')$ (condition Rho-her). Thus $w' \models P$.

Case. $A = \top$ is trivial

Case. $A = \bot$
Suppose $w \models \bot$ and $w' \geq w$.
To show: $w' \models \bot$.
By assumption $w \in F$. Hence by (F-her), $w' \in F$. Thus $w' \models \bot$.

Case. $A = A_1 \wedge A_2$
Suppose $w \models A_1 \wedge A_2$ and $w' \geq w$.
To show: $w' \models A_1 \wedge A_2$.
By assumption, $w \models A_1$ and $w \models A_2$. By i.h., $w' \models A_1$ and $w' \models A_2$. Thus by definition $w' \models A_1 \wedge A_2$, as required.

Case. $A = A_1 \vee A_2$
Suppose $w \models A_1 \vee A_2$ and $w' \geq w$.
To show: $w' \models A_1 \vee A_2$.
By assumption, $w \models A_1$ or $w \models A_2$. Let us take the case $w \models A_1$ (the other case is symmetric). By i.h., $w' \models A_1$. Thus by definition $w' \models A_1 \vee A_2$, as required.

Case. $A = A_1 \supset A_2$
Suppose $w \models A_1 \supset A_2$ and $w' \geq w$.
To show: $w' \models A_1 \supset A_2$.
Suppose $w'' \geq w'$.
To show: $w'' \models A_1$ implies $w'' \models A_2$.
By transitivity of $\geq$, $w'' \geq w$. By definition of $w \models A_1 \supset A_2$, $w'' \models A_1$ implies $w'' \models A_2$ as required.

Case. $A = K \mathsf{~says~} B$
Suppose $w \models K \mathsf{~says~} B$ and $w' \geq w$.
To show: $w' \models K \mathsf{~says~} B$.
By assumption $w \models K \mathsf{~says~} B$, either $w \in F$ or $w \leq w''' \sqsubseteq_K w''$ implies $w'' \models A$. If $w \in F$, by (F-her), $w' \in F$. Hence $w' \models K \mathsf{~says~} B$. Otherwise, pick any w_4 and w_5 such that $w' \leq w_4 \sqsubseteq_K w_5$. Clearly, by transitivity of $\leq$, we also have $w \leq w_4 \sqsubseteq_K w_5$. Thus by the assumption $w_5 \models A$ as required.

□

Lemma D.2 (Falsehood)**.** *If $w \models \bot$, then for any proposition A, $w \models A$.*

Proof. By induction on A.

Case. $A = P$ (A is atomic)
Suppose $w \models \bot$
To show: $w \models P$.
By assumption, $w \in F$. By (F-univ), $P \in \rho(w)$. Thus $w \models P$.

Case. $A = \top$ is trivial.

Case. $A = \bot$ is trivial.

Case. $A = A_1 \wedge A_2$.
Suppose $w \models \bot$. By i.h. $w \models A_1$ and $w \models A_2$. Hence by definition of satisfaction, $w \models A_1 \wedge A_2$.

Case. $A = A_1 \vee A_2$.
Suppose $w \models \bot$. By i.h. $w \models A_1$. Hence by definition of satisfaction, $w \models A_1 \vee A_2$.

Case. $A = A_1 \supset A_2$.
Suppose $w \models \bot$. Choose any $w' \geq w$. We want to show that $w' \models A$ implies $w' \models B$. By Lemma D.1, $w' \models \bot$. Hence $w' \models B$ by i.h., and in particular, $w' \models A$ implies $w' \models B$.

Case. $A = K \mathsf{~says~} B$.
Suppose $w \models \bot$. Thus by definition, $w \in F$. Hence by definition, $w \models K \mathsf{~says~} B$.

□

Theorem D.3 (Soundness)**.** *If $\Gamma \xrightarrow{K} A$, then for each Kripke model M, $M \models^K \overline{\Gamma} \supset A$.*

Proof. We induct on the derivation of $\Gamma \xrightarrow{K} A$.

Case. $$\frac{P \text{ atomic}}{\Gamma, P \xrightarrow{K} P}\text{init}$$

Pick any w such that $K \in \theta(w)$. We want to show that $w \models (\overline{\Gamma} \wedge P) \supset P$. So pick any $w' \geq w$. It suffices to show that $w' \models \overline{\Gamma} \wedge P$ implies $w' \models P$. But if we assume that $w' \models \overline{\Gamma} \wedge P$, then $w' \models P$ follows by definition of satisfaction.

Case. $$\frac{\Gamma, K \textsf{ claims } A, A \xrightarrow{K'} C \qquad K \succeq K'}{\Gamma, K \textsf{ claims } A \xrightarrow{K'} C}\text{claims}$$

Pick any w ($K' \in \theta(w)$)

To show: $w \models (\overline{\Gamma} \wedge K \textsf{ says } A) \supset C$

1. Assume any $w' \geq w$

   Suffices to show: $w' \models \overline{\Gamma} \wedge K \textsf{ says } A$ implies $w' \models C$
2. Assume: $w' \models \overline{\Gamma}$ and $w' \models K \textsf{ says } A$
3. Suffices to show: $w' \models C$
4. From i.h.: $w' \models (\overline{\Gamma} \wedge (K \textsf{ says } A) \wedge A) \supset C$. (We can apply i.h. since $K' \in \theta(w)$ and $w \leq w'$ imply $K' \in \theta(w')$)
5. By definition on (4): $w' \models \overline{\Gamma}$ and $w' \models K \textsf{ says } A$ and $w' \models A$ implies $w' \models C$
6. From (2), $w' \models K \textsf{ says } A$. Also, $K' \in \theta(w')$. Hence, by (View-close), $K \in \theta(w')$. By (Imp-refl) and (Mod-refl), $w' \leq w' \sqsubseteq_K w'$. By definition of satisfaction, either $w' \in F$, or $w' \models A$. In the former case, $w' \models A$ by Lemma D.2. Thus $w' \models A$.
7. From (2), (6) and (5) , $w' \models C$ as required in (3).

Case. $$\frac{\Gamma|_K \xrightarrow{K} A}{\Gamma \xrightarrow{K'} K \textsf{ says } A}\textsf{saysR}$$

Pick any w such that $K' \in \theta(w)$.

To show: $w \models \overline{\Gamma} \supset K \textsf{ says } A$.

1. Assume any $w' \geq w$

   Suffices to show: $w' \models \overline{\Gamma}$ implies $w' \models K \textsf{ says } A$
2. Assume $w' \models \overline{\Gamma}$.
3. Suffices to show: $w' \models K \textsf{ says } A$
4. Assume $w' \notin F$, and pick any w'', w''' such that $w' \leq w'' \sqsubseteq_K w'''$ (Note: $K \in \theta(w''')$).
5. Suffices to show: $w''' \models A$
6. Let $\Gamma|_K = K_1 \textsf{ claims } B_1, \ldots, K_n \textsf{ claims } B_n$. (Note: $K_i \succeq K$ for each i)
7. By i.h., $w''' \models (K_1 \textsf{ says } B_1 \wedge \ldots \wedge K_n \textsf{ says } B_n) \supset A$ (we can apply i.h. because $K \in \theta(w'''))$

8. By definition on (7), $w''' \models K_i \textsf{ says } B_i$ for all i implies $w''' \models A$
9. From (5) and (7), suffices to show that $w''' \models K_i \textsf{ says } B_i$ for each i.
10. Choose any $w''' \leq w_4 \sqsubseteq_{K_i} w_5$. Suffices to show: $w_5 \models B_i$. (Note: $K \in \theta(w_4)$ by (View-closure))
11. From (4) and (10), we have $w'' \sqsubseteq_K w''' \leq w_4$. By (Commutativity), $w'' \sqsubseteq_K w_4$. By (Mod-closure), $w'' \sqsubseteq_{K_i} w_4$.
12. We now obtain $w' \leq w'' \sqsubseteq_{K_i} w_4 \sqsubseteq_{K_i} w_5$.
13. From (12), and (Mod-trans), $w' \leq w'' \sqsubseteq_{K_i} w_5$.
14. Since $w' \models K \textsf{ says } B_i$ (assumption 2), and $w' \notin F$ (assumption 4), by definition of satisfaction, $w_5 \models B_i$, as required in (10).

Case. $\dfrac{\Gamma, K \textsf{ says } A, K \textsf{ claims } A \xrightarrow{K'} C}{\Gamma, K \textsf{ says } A \xrightarrow{K'} C}\textsf{saysL}$

We want to show that if $w^{K'}$, then $w \models (\overline{\Gamma} \wedge K \textsf{ says } A) \supset C$. Equivalently, for any $w' \geq w$, $w' \models \overline{\Gamma} \wedge K \textsf{ says } A$ implies $w' \models A$. By i.h., $w' \models \overline{\Gamma} \wedge K \textsf{ says } A \wedge K \textsf{ says } A$ implies $w' \models A$. However, $w' \models \overline{\Gamma} \wedge K \textsf{ says } A$ and $w' \models \overline{\Gamma} \wedge K \textsf{ says } A \wedge K \textsf{ says } A$ are equivalent by definition.

Case. $\dfrac{\Gamma \xrightarrow{K} A \qquad \Gamma \xrightarrow{K} B}{\Gamma \xrightarrow{K} A \wedge B}\wedge\mathrm{R}$

Suppose w^K is a world. We want to show that $w \models \overline{\Gamma} \supset (A \wedge B)$. Pick any $w' \geq w$ and assume that $w' \models \overline{\Gamma}$. It suffices to show that $w' \models A \wedge B$, or equivalently that $w' \models A$ and that $w' \models B$. This follows immediately by i.h. (since $K \in \theta(w')$)

Case. $\dfrac{\Gamma, A \wedge B, A, B \xrightarrow{K} C}{\Gamma, A \wedge B \xrightarrow{K} C}\wedge\mathrm{L}$

Suppose w^K is a world. Pick any $w' \geq w$. It suffices to show that $w' \models \overline{\Gamma} \wedge A \wedge B$ implies that $w' \models C$. This follows immediately by the i.h. (since $K \in \theta(w')$).

Case. $\dfrac{\Gamma \xrightarrow{K} A}{\Gamma \xrightarrow{K} A \vee B}\vee\mathrm{R}_1$

Suppose w^K is a world. Pick any $w' \geq w$. It suffices to show that $w' \models \overline{\Gamma}$ implies that $w' \models A \vee B$. Assume $w' \models \overline{\Gamma}$. By i.h., $w' \models \overline{\Gamma}$ implies $w' \models A$. Hence $w' \models A$. By definition of satisfaction, $w' \models A \vee B$ as required.

Case. $\dfrac{\Gamma \xrightarrow{K} B}{\Gamma \xrightarrow{K} A \vee B}\vee\mathrm{R}_2$

Similar to the previous case.

Case. $\dfrac{\Gamma, A \vee B, A \xrightarrow{K} C \qquad \Gamma, A \vee B, B \xrightarrow{K} C}{\Gamma, A \vee B \xrightarrow{K} C}\vee\mathrm{L}$

Suppose w^K is a world. Pick any $w' \geq w$. We want to show that $w' \models \overline{\Gamma} \wedge (A \vee B)$

implies $w' \models C$. Suppose that $w' \models \overline{\Gamma} \wedge (A \vee B)$. By definition, $w' \models \overline{\Gamma}$ and either $w' \models A$ or $w' \models B$. Suppose that $w' \models A$ (the other case is similar). By i.h. on first premise, $w' \models \overline{\Gamma}$ and $w' \models A$ imply $w' \models C$. It follows immediately that $w' \models C$ as required.

Case. $\dfrac{}{\Gamma \xrightarrow{K} \top}\top\mathrm{R}$

Pick any w^K. We want to show for any $w' \geq w$ that $w' \models \overline{\Gamma}$ implies $w' \models \top$. However, $w' \models \top$ is always true by definition.

Case. $\dfrac{}{\Gamma, \bot \xrightarrow{K} C}\bot\mathrm{L}$

Pick any w^K. We want to show for any $w' \geq w$ that $w' \models \overline{\Gamma} \wedge \bot$ implies $w' \models C$. Assume that $w' \models \overline{\Gamma} \wedge \bot$. In particular, $w' \models \bot$. By Lemma D.2, $w' \models C$, as required.

Case. $\dfrac{\Gamma, A \xrightarrow{K} B}{\Gamma \xrightarrow{K} A \supset B}\supset\mathrm{R}$

Pick any w^K, and any $w' \geq w$. It suffices to show that $w' \models \overline{\Gamma}$ implies $w' \models A \supset B$. To show this pick any $w'' \geq w'$, assume that $w'' \models A$ and observe that it suffices to show that $w'' \models B$. Now from Lemma D.1, it follows that $w'' \models \overline{\Gamma}$. Hence $w'' \models \overline{\Gamma} \wedge A$. From i.h., $w'' \models B$ as required.

Case. $\dfrac{\Gamma, A \supset B \xrightarrow{K} A \qquad \Gamma, A \supset B, B \xrightarrow{K} C}{\Gamma, A \supset B \xrightarrow{K} C}\supset\mathrm{L}$

Pick any w^K and $w' \geq w$. We want to show that $w' \models \overline{\Gamma} \wedge (A \supset B)$ implies $w' \models C$. Assume that $w' \models \overline{\Gamma}$ and that $w' \models A \supset B$. From i.h. (1) it follows that $w' \models A$. Hence $w' \models B$. Now from i.h. (2), $w' \models C$ as required.

$\square$

D.2 Canonical Kripke Model and Completeness

In this section, we provide proofs of Lemmas and Theorems from Section 4.1.

Lemma D.4 (Canonical Model; Lemma 4.8)**.** *The model constructed in Definition 4.7 is a Kripke model for DTL$_0$, i.e., it satisfies all conditions of Definition 4.1.*

Proof. We verify all the conditions from Definition 4.1.

- (View-closure) Suppose $K \in \theta(\Gamma, S) = S$. Now by (Prin-closure), there is a principal K_0 such that $S = \{K \mid K \succeq K_0\}$. It follows that $K \succeq K_0$. Now suppose $K' \succeq K$. By the fact that $\succeq$ is a pre-order, $K' \succeq K_0$. Thus $K' \in S = \theta(\Gamma, S)$, as required.

- (Imp-mon) By definition, $(\Gamma, S) \leq (\Gamma', S')$ implies $S \subseteq S'$, i.e., $\theta(\Gamma, S) \subseteq \theta(\Gamma', S')$.

- (Imp-refl) and (Imp-trans) follow by definition of $\leq$ in the canonical model.

- (Mod-refl) Let $w = (\Gamma, S)$ and $K \in S$. We want to show that $(\Gamma, S) \sqsubseteq_K (\Gamma, S)$. For this, we need to show that $K\ \mathsf{says}\ A \in \Gamma$ implies $A \in \Gamma$. This follows from the condition (Fact-closure) because $K\ \mathsf{says}\ A \xrightarrow{K} A$ in the sequent calculus.

- (Mod-trans) Let $(\Gamma_1, S_1) \sqsubseteq_K (\Gamma_2, S_2) \sqsubseteq_K (\Gamma_3, S_3)$. We want to show that $K\ \mathsf{says}\ A \in S_1$ implies $A \in S_3$. By condition (Prin-closure), there is at least one element in S_1, say K_1. Now observe that $K\ \mathsf{says}\ A \xrightarrow{K_1} K\ \mathsf{says}\ K\ \mathsf{says}\ A$. Hence $K\ \mathsf{says}\ A \in \Gamma_1$ implies (by Fact-closure) that $K\ \mathsf{says}\ K\ \mathsf{says}\ A \in \Gamma_1$, which implies (by definition of $\sqsubseteq_K$) that $K\ \mathsf{says}\ A \in \Gamma_2$, which further implies that $A \in \Gamma_3$.

- (Mod-closure) Let $(\Gamma_1, S_1) \sqsubseteq_K (\Gamma_2, S_2)$ and $K' \succeq K$. We want to show $(\Gamma_1, S_1) \sqsubseteq_{K'} (\Gamma_2, S_2)$. Clearly, $K' \in S_2$ because $K \in S_2$. Pick any $K'\ \mathsf{says}\ B \in \Gamma_1$. We need to show $B \in \Gamma_2$. However, $K'\ \mathsf{says}\ B \xrightarrow{K_0} K\ \mathsf{says}\ B$ for any $K_0 \in S_1$. Thus by (Fact-closure), $K\ \mathsf{says}\ B \in \Gamma_1$. Hence by definition of $\sqsubseteq_K$, $B \in \Gamma_2$ as required.

- (Rho-her) Let $P \in \rho(\Gamma_1, S_1)$ (i.e., $P \in \Gamma_1$) and $(\Gamma_1, S_1) \leq (\Gamma_2, S_2)$. By definition of $\leq$ in the canonical model, $\Gamma_1 \subseteq \Gamma_2$. Thus $P \in \Gamma_2$, or equivalently, $P \in \rho(\Gamma_2, S_2)$.

- (F-her) Let $(\Gamma_1, S_1) \in F$ (i.e., $\perp \in \Gamma_1$) and $(\Gamma_1, S_1) \leq (\Gamma_2, S_2)$. By definition of $\leq$ in the canonical model, $\Gamma_1 \subseteq \Gamma_2$. Thus $\perp \in \Gamma_2$, or equivalently, $(\Gamma_2, S_2) \in F$.

- (F-univ) Let $(\Gamma, S) \in F$. By definition, $\perp \in \Gamma$. By condition (Prin-closure), there is at least one principal, say K, in S. Also, $\perp \xrightarrow{K} P$ in the sequent calculus. Therefore, by condition (Fact-closure), $P \in \Gamma$, or equivalently, $P \in \rho(\Gamma, S)$.

- (Commutativity) Suppose $(\Gamma_1, S_1) \sqsubseteq_K (\Gamma_2, S_2) \leq (\Gamma_3, S_3)$. We want to show that $(\Gamma_1, S_1) \sqsubseteq_K (\Gamma_3, S_3)$. Since $K \in S_2$ and $S_2 \subseteq S_3$, $K \in S_3$. Also, $K\ \mathsf{says}\ A \in \Gamma_1$ implies (by definition of $\sqsubseteq_K$) that $A \in \Gamma_2$ which in turn implies $A \in \Gamma_3$ (since $\Gamma_2 \subseteq \Gamma_3$).

□

Lemma D.5 (Consistent Extensions; Lemma 4.9)**.** *Let* (Γ, S) *be an A consistent theory. Then there is an A consistent prime theory* (Γ^*, S) *such that* $\Gamma \subseteq \Gamma^*$*.*

Proof. We use Zorn's lemma. Let us define the set $\mathcal{S}$ of theories as follows: $\mathcal{S} = \{(\Gamma', S) \mid \Gamma' \supseteq \Gamma,\ \forall K \in S.\ \Gamma' \stackrel{K}{\not\longrightarrow} A\}$. We make the set a partial order by defining $(\Gamma', S) \leq (\Gamma'', S)$ if $\Gamma' \subseteq \Gamma''$. Clearly $\mathcal{S}$ is non-empty since $(\Gamma, S) \in \mathcal{S}$. Now take any chain $(\Gamma_1, S) \leq (\Gamma_2, S) \leq \ldots$ in $\mathcal{S}$. Clearly $(\cup\Gamma_i, S)$ is an upper bound on this chain. We show that $(\cup\Gamma_i, S) \in \mathcal{S}$. First, clearly $\cup\Gamma_i \supseteq \Gamma$ since each $\Gamma_i \supseteq \Gamma$. Second, $\cup\Gamma_i \stackrel{K}{\not\longrightarrow} A$ for any $K \in S$. To see this, assume (for the sake of contradiction) that $\cup\Gamma_i \xrightarrow{K} A$ for some $K \in S$. Then there is a finite subset Γ' of $\cup\Gamma_i$ such that $\Gamma' \xrightarrow{K} A$. Since Γ' is finite, there must be some n such that $\Gamma' \subseteq \Gamma_n$. Clearly then $\Gamma_n \xrightarrow{K} A$, thus violating the fact that $(\Gamma_n, S) \in \mathcal{S}$. Hence, $\cup\Gamma_i \stackrel{K}{\not\longrightarrow} A$. And therefore, $(\cup\Gamma_i, S) \in \mathcal{S}$.

By Zorn's lemma, $\mathcal{S}$ has a maximal element. Let this element be (Γ^*, S). By definition of $\mathcal{S}$, $\Gamma^* \stackrel{K}{\not\longrightarrow} A$ for any $K \in S$, so that (Γ^*, S) is A consistent. We now show that (Γ^*, S) is a prime theory. To do this, we verify the (Fact-closure) and (Primality) conditions. (The condition (Prin-closure) holds because we assume that S is a filter).

- (Fact-closure) Suppose for the sake of contradiction that $\Gamma^* \xrightarrow{K} C$ for some $K \in S$, but $C \notin \Gamma^*$. Let K_0 be a minimum element of S (this exists because S is a filter). Clearly $\Gamma^*, C \not\xrightarrow{K_0} A$. (If not, then $\Gamma^*, C \xrightarrow{K_0} A$ and $\Gamma^* \xrightarrow{K} C$ would imply $\Gamma^* \xrightarrow{K_0} A$ by Theorems B.1 and B.2, thus contradicting the A consistency of (Γ^*, S).) It follows that for any $K \in S$, $\Gamma^*, C \not\xrightarrow{K} A$. Thus $(\Gamma^* \cup \{C\}, S) \in \mathcal{S}$. This contradicts the maximality of (Γ^*, S).

- (Primality) Suppose for the sake of contradiction that $B \vee C \in \Gamma^*$ and $B, C \notin \Gamma^*$. Consider the theories $(\Gamma^* \cup \{B\}, S)$ and $(\Gamma^* \cup \{C\}, S)$. We claim that at least one of these is A consistent. Suppose on the contrary that both are A inconsistent. Then $\Gamma^*, B \xrightarrow{K_1} A$ and $\Gamma^*, C \xrightarrow{K_2} A$ for some $K_1, K_2 \in S$. Thus $\Gamma^*, B \vee C \xrightarrow{K_0} A$, where K_0 is a least element of S. Further, since $B \vee C \in \Gamma^*$, we would obtain $\Gamma^* \xrightarrow{K_0} A$, thus violating the A consistency of (Γ^*, S). Hence at least one of the theories $(\Gamma^* \cup \{B\}, S)$ and $(\Gamma^* \cup \{C\}, S)$ is A consistent. Assume without loss of generality that $(\Gamma^* \cup \{B\}, S)$ is A consistent. Then clearly, $(\Gamma^* \cup \{B\}, S) \in \mathcal{S}$, which violates the maximality of (Γ^*, S). Thus at least one of B and C must be in Γ^*, as required.

□

Lemma D.6 (Satisfaction; Lemma 4.10)**.** *For each formula A, and each prime theory (Γ, S), it is the case that $(\Gamma, S) \models A$ in the canonical model iff $A \in \Gamma$.*

Proof. We induct on A.

Case. $A = P$ (A is atomic).

$(\Gamma, S) \models P$ iff $P \in \rho(\Gamma, S)$ iff $P \in \Gamma$.

Case. $A = B \wedge C$.

Suppose $B \wedge C \in \Gamma$. We want to show that $(\Gamma, S) \models B \wedge C$. By (Fact-closure) on the theory (Γ, S), $B \in \Gamma$ and $C \in \Gamma$. Hence by the i.h., $(\Gamma, S) \models B$ and $(\Gamma, S) \models C$. It follows then that $(\Gamma, S) \models B \wedge C$.

Conversely suppose that $(\Gamma, S) \models B \wedge C$. By definition $(\Gamma, S) \models B$ and $(\Gamma, S) \models C$. By i.h., $B, C \in \Gamma$. By (Fact-closure), $B \wedge C \in \Gamma$.

Case. $A = B \vee C$.

Suppose $B \vee C \in \Gamma$. We want to show that $(\Gamma, S) \models B \vee C$. By the (Primality) condition on (Γ, S), either $B \in \Gamma$ or $C \in \Gamma$. Assume the former (the other case is similar). Then by i.h., $(\Gamma, S) \models B$. Hence by definition, $(\Gamma, S) \models B \vee C$.

Conversely, suppose that $(\Gamma, S) \models B \vee C$. By definition, either $(\Gamma, S) \models B$ or $(\Gamma, S) \models C$. Assume the former (the other case is similar). Then by i.h., $B \in \Gamma$. Hence by (Fact-closure), $B \vee C \in \Gamma$.

Case. $A = \top$.

Suppose $\top \in \Gamma$. Then trivially, $(\Gamma, S) \models \top$.

Conversely, suppose $(\Gamma, S) \models \top$. Since $\Gamma \xrightarrow{K} \top$ for any K, $\top \in \Gamma$ by (Fact-closure).

Case. $A = \bot$.

$(\Gamma, S) \models \bot$ iff $(\Gamma, S) \in F$ iff $\bot \in \Gamma$.

Case. $A = B \supset C$

Suppose $B \supset C \in \Gamma$. Pick any $(\Gamma', S') \geq (\Gamma, S)$. We want to show that $(\Gamma', S') \models B$ implies $(\Gamma', S') \models C$. Assume $(\Gamma', S') \models B$. By i.h., $B \in \Gamma'$. Also, by definition, $\Gamma' \supseteq \Gamma$. Hence, $B \supset C \in \Gamma'$. Clearly $B, B \supset C \xrightarrow{K} C$ for any $K \in S'$. Thus by (Fact-closure) on (Γ', S'), it must be the case that $C \in \Gamma'$. By i.h., $(\Gamma', S') \models C$, as required.

Conversely, suppose that $(\Gamma, S) \models B \supset C$. We want to show that $B \supset C \in \Gamma$. Assume for the sake of contradiction that $B \supset C \notin \Gamma$, and pick any $K \in S$. Due to (Fact-closure), it must be the case that $\Gamma \stackrel{K}{\not\longrightarrow} B \supset C$. It follows immediately (from basic properties of the sequent calculus) that $\Gamma, B \stackrel{K}{\not\longrightarrow} B \supset C$. Thus $(\Gamma \cup \{B\}, S)$ is $B \supset C$ consistent. By Lemma D.5, there is a prime theory $w = (\Gamma \cup \{B\} \cup \Gamma', S)$ which is $B \supset C$ consistent. Now by i.h., $w \models B$. Also, since $w \geq (\Gamma, S)$, and $(\Gamma, S) \models B \supset C$, we obtain $w \models C$. By i.h., $C \in \Gamma \cup \{B\} \cup \Gamma'$. Since $C \xrightarrow{K} B \supset C$ for any K, it follows by (Fact-closure) on w that $B \supset C \in \Gamma \cup \{B\} \cup \Gamma'$, which violates the $B \supset C$ consistency of $(\Gamma \cup \{B\} \cup \Gamma', S)$. It follows therefore, that $B \supset C \in \Gamma$, as required.

Case. $A = K$ says B

Suppose K says $B \in \Gamma$. We want to show that $(\Gamma, S) \models K$ says B. So pick any sequence: $(\Gamma, S) \leq (\Gamma', S') \sqsubseteq_K (\Gamma'', S'')$, where $K \in S''$. Since $\Gamma \subseteq \Gamma'$, K says $B \in \Gamma'$. By definition of $\sqsubseteq_K$ in canonical models, $B \in \Gamma''$. Hence by i.h., $(\Gamma'', S'') \models B$. Since $\Gamma', \Gamma'', S', S''$ are arbitrary, by definition of satisfaction it follows that $(\Gamma, S) \models K$ says B.

Conversely, suppose that $(\Gamma, S) \models K$ says B. We want to show that K says $B \in \Gamma$. If $\bot \in \Gamma$, this is trivial due to the closure condition. Hence we may assume that $\bot \notin \Gamma$. Let $S_K = \{K' \mid K' \succeq K\}$. Now consider the theory $(\Gamma|_K, S_K)$[9]. We claim that this theory is not B consistent. Suppose on the contrary that it is B consistent. Then by Lemma D.5, there is a larger prime theory (Γ', S_K) that is B consistent ($\Gamma' \supseteq \Gamma|_K$). Observe that $(\Gamma, S) \leq (\Gamma, S) \sqsubseteq_K (\Gamma', S_K)$. The first relation is trivial. To prove that $(\Gamma, S) \sqsubseteq_K (\Gamma', S_K)$, pick any K' says $C \in \Gamma$, where $K' \succeq K$. We will show that $C \in \Gamma'$. Since K' says $C \in \Gamma$, K' says $C \in \Gamma|_K$. Hence, K' says $C \in \Gamma'$. Now observe that K' says $C \xrightarrow{K} C$. So by (Fact-closure) on (Γ', S_K), $C \in \Gamma'$. Hence $(\Gamma, S) \leq (\Gamma, S) \sqsubseteq_K (\Gamma', S_K)$.

Next, from the assumption that $(\Gamma, S) \models K$ says B and the fact that $\bot \notin \Gamma$ (so that (Γ, S) is not fallible), we must have $(\Gamma', S_K) \models B$. By i.h., $B \in \Gamma'$. This immediately violates the fact that (Γ', S_K) is B consistent. Thus $(\Gamma|_K, S_K)$ is not B consistent. Therefore there is some $K' \in S_K$ such that $\Gamma|_K \xrightarrow{K'} B$. Since $K' \succeq K$, it follows from Theorem B.1 that $\Gamma|_K \xrightarrow{K} B$. Thus $\Gamma \xrightarrow{K''} K$ says B for any $K'' \in S$. Thus by (Fact-closure) on (Γ, S), K says $B \in \Gamma$.

[9] $\Gamma|_K$ is defined here as $\{(K'$ says $A) \in \Gamma \mid K' \succeq K\}$

$\square$

Theorem D.7 (Soundness and Completeness; Theorem 4.3). *$\cdot \xrightarrow{K} A$ if and only if for each Kripke model M, $M \models^K A$.*

Proof. Suppose $\cdot \xrightarrow{K} A$. Then by Theorem D.3, $M \models^K \top \supset A$. Hence, $M \models^K A$. Conversely, suppose that for each model M, $M \models^K A$. Then, in particular for each w^K in the canonical model $w \models A$. By Theorem 4.11, $\cdot \xrightarrow{K} A$ (else there must be a world w^K such that $w \not\models A$). $\square$

E Proofs from Section 5.1

In this appendix we prove that the translation from DTL_0 to CS4^m is correct (Theorem 5.2). First, we develop the axiomatic system for CS4^m.

E.1 The Axiomatic System for CS4^m

In Section 5.1, we listed the axioms and rules of CS4^m that are specific to the modality $\square_K$. Below we list *all* the rules and axioms of CS4^m.

$$\frac{\vdash A}{\vdash \square_K A}\text{nec} \qquad \frac{\vdash A \supset B \qquad \vdash A}{\vdash B}\text{mp} \qquad \frac{A \text{ is an axiom}}{\vdash A}\text{ax}$$

Axioms:

$(\square_K(A \supset B)) \supset ((\square_K A) \supset (\square_K B))$	(K)
$(\square_K A) \supset \square_K \square_K A$	(4)
$(\square_K A) \supset A$	(T)
$(\square_K A) \supset \square_{K'} A$ if $K \succeq K'$	(S)
$A \supset (B \supset A)$	(imp1)
$(A \supset B) \supset ((A \supset (B \supset C)) \supset (A \supset C))$	(imp2)
$A \supset (B \supset (A \wedge B))$	(conj1)
$(A \wedge B) \supset A$	(conj2)
$(A \wedge B) \supset B$	(conj3)
$A \supset (A \vee B)$	(disj1)
$B \supset (A \vee B)$	(disj2)
$(A \supset C) \supset ((B \supset C) \supset ((A \vee B) \supset C))$	(disj3)
$\top$	(true)
$\perp \supset A$	(false)

Next, we generalize the axiomatic system, adding hypothetical reasoning, as we did for the axiomatic system of DTL_0 (Appendix C). We write $\Gamma \vdash_G A$ to mean that A follows from the formulas in Γ. The rules of deduction are:

$$\frac{}{\Gamma, A \vdash_G A}\text{use} \qquad \frac{\cdot \vdash_G A}{\Gamma \vdash_G \square_K A}\text{nec} \qquad \frac{\Gamma \vdash_G A \supset B \qquad \Gamma \vdash_G A}{\Gamma \vdash_G B}\text{mp} \qquad \frac{A \text{ is an axiom}}{\Gamma \vdash_G A}\text{ax}$$

As for DTL_0, we prove some elementary properties for the generalized system of CS4^m, and show also that the generalized system and axiomatic system are equivalent.

Lemma E.1 (Basic properties). *The following hold.*

1. *(Weakening)* $\Gamma \vdash_G A$ *implies* $\Gamma, \Gamma' \vdash_G A$
2. *(Substitution)* $\Gamma \vdash_G A$ *and* $\Gamma, A \vdash_G B$ *imply* $\Gamma \vdash_G B$

Proof. Exactly as for DTL_0 in Lemma C.1, since the proof does not rely on the specific axioms used. □

Theorem E.2 (Deduction). *The following hold.*

1. $\Gamma \vdash_G A \supset B$ *implies* $\Gamma, A \vdash_G B$
2. $\Gamma, A \vdash_G B$ *implies* $\Gamma \vdash_G A \supset B$

Proof. Exactly as for DTL_0 in Theorem C.2. The proof does not rely on the axiom (C), which is the only axiom present in DTL_0 that is not present in $\mathrm{CS4}^m$. □

Theorem E.3 (G iff Axiomatic). $\vdash A$ *if and only if* $\cdot \vdash_G A$

Proof. In each direction by straightforward induction on the given derivation. □

E.2 Proof of Soundness

Next we prove soundness of the translation, $\cdot \xrightarrow{K} A$ in DTL_0 implies $\vdash \mathcal{O} \supset (K \supset \ulcorner A \urcorner)$ in $\mathrm{CS4}^m$. Instead of establishing exactly this statement, we modify it slightly to make our induction easier.

Lemma E.4 (Soundness of Translation). *If* $\vdash_H A$ *in* DTL_0*'s axiomatic system, then* $\mathcal{O} \vdash_G \ulcorner A \urcorner$ *in* $CS4^m$*'s generalized axiomatic system.*

Proof. We induct on the given derivation of $\vdash_H A$, case analyzing the last rule. (We remind the reader that the rules for the judgment $\vdash_H$ are listed in Appendix C.)

Case. $\dfrac{\vdash_H A}{\vdash_H K \textsf{ says } A}\text{nec}$

Let $\mathcal{O} = \Box_\ell(K_{1A} \supset K_{1B}), \ldots, \Box_\ell(K_{nA} \supset K_{nB})$.

1. $\mathcal{O} \vdash_G \ulcorner A \urcorner$ (i.h. on premise)
2. $\mathcal{O}, K \vdash_G \ulcorner A \urcorner$ (Weakening, Lemma E.1.1)
3. $\cdot \vdash_G \mathcal{O} \supset (K \supset \ulcorner A \urcorner)$ (Theorem E.2)
4. $\cdot \vdash_G \Box_K(\mathcal{O} \supset (K \supset \ulcorner A \urcorner))$ (Rule (nec))
5. $\cdot \vdash_G (\Box_K \mathcal{O}) \supset \Box_K(K \supset \ulcorner A \urcorner)$ (Rule (ax), (K), and rule (mp))
6. $\Box_K \Box_\ell (K_{1A} \supset K_{1B}), \ldots, \Box_K \Box_\ell (K_{nA} \supset K_{nB}) \vdash_G \Box_K(K \supset \ulcorner A \urcorner)$ (Theorem E.2)
7. $\cdot \vdash_G (\Box_\ell(K_{iA} \supset K_{iB})) \supset \Box_\ell \Box_\ell (K_{iA} \supset K_{iB})$ (Rule (ax) and (4))
8. $\Box_\ell(K_{iA} \supset K_{iB}) \vdash_G \Box_\ell \Box_\ell (K_{iA} \supset K_{iB})$ (Theorem E.2)

9. $\cdot \vdash_G (\Box_\ell \Box_\ell (K_{iA} \supset K_{iB})) \supset \Box_K \Box_\ell (K_{iA} \supset K_{iB})$ (Rule (ax) and (S))

10. $\Box_\ell \Box_\ell (K_{iA} \supset K_{iB}) \vdash_G \Box_K \Box_\ell (K_{iA} \supset K_{iB})$ (Theorem E.2)

11. $\Box_\ell (K_{iA} \supset K_{iB}) \vdash_G \Box_K \Box_\ell (K_{iA} \supset K_{iB})$ (Lemma E.1.2 on 8 and 10)

12. $\mathcal{O} \vdash_G \Box_K (K \supset \ulcorner A \urcorner)$ (Lemma E.1.2 on 11 and 6)

Case. $\dfrac{\vdash_H A \supset B \qquad \vdash_H A}{\vdash_H B}\text{mp}$

1. $\mathcal{O} \vdash_G \ulcorner A \urcorner \supset \ulcorner B \urcorner$ (i.h. on 1st premise)
2. $\mathcal{O} \vdash_G \ulcorner A \urcorner$ (i.h. on 2nd premise)
3. $\mathcal{O} \vdash_G \ulcorner B \urcorner$ (Rule (mp))

Case. $\dfrac{A \text{ is an axiom}}{\vdash_H A}\text{ax}$

We case analyze the axioms.

Case. (Axiom K) $A = (K' \textsf{ says } (A' \supset B')) \supset ((K' \textsf{ says } A') \supset (K' \textsf{ says } B'))$

1. $\mathcal{O} \vdash_G (\Box_{K'}((K' \supset \ulcorner A' \urcorner) \supset (K' \supset \ulcorner B' \urcorner))) \supset ((\Box_{K'}(K' \supset \ulcorner A' \urcorner)) \supset (\Box_{K'}(K' \supset \ulcorner B' \urcorner)))$ (Rule (ax) and (K))
2. $\mathcal{O}, \Box_{K'}((K' \supset \ulcorner A' \urcorner) \supset (K' \supset \ulcorner B' \urcorner)) \vdash_G ((\Box_{K'}(K' \supset \ulcorner A' \urcorner)) \supset (\Box_{K'}(K' \supset \ulcorner B' \urcorner)))$ (Theorem E.2)
3. $\cdot \vdash_G (K' \supset (\ulcorner A' \urcorner \supset \ulcorner B' \urcorner)) \supset ((K' \supset \ulcorner A' \urcorner) \supset (K' \supset \ulcorner B' \urcorner))$
   (Basic propositional theorem)
4. $\cdot \vdash_G (\Box_{K'}(K' \supset (\ulcorner A' \urcorner \supset \ulcorner B' \urcorner))) \supset \Box_{K'}((K' \supset \ulcorner A' \urcorner) \supset (K' \supset \ulcorner B' \urcorner))$
   (Rule (ax), (K) and rule (mp))
5. $\Box_{K'}(K' \supset (\ulcorner A' \urcorner \supset \ulcorner B' \urcorner)) \vdash_G \Box_{K'}((K' \supset \ulcorner A' \urcorner) \supset (K' \supset \ulcorner B' \urcorner))$
   (Theorem E.2)
6. $\mathcal{O}, \Box_{K'}(K' \supset (\ulcorner A' \urcorner \supset \ulcorner B' \urcorner)) \vdash_G ((\Box_{K'}(K' \supset \ulcorner A' \urcorner)) \supset (\Box_{K'}(K' \supset \ulcorner B' \urcorner)))$
   (Lemma E.1.2 on 5 and 2)
7. $\mathcal{O} \vdash_G (\Box_{K'}(K' \supset (\ulcorner A' \urcorner \supset \ulcorner B' \urcorner))) \supset ((\Box_{K'}(K' \supset \ulcorner A' \urcorner)) \supset (\Box_{K'}(K' \supset \ulcorner B' \urcorner)))$
   (Theorem E.2)

Case. (Axiom 4) $A = (K' \textsf{ says } A') \supset K' \textsf{ says } K' \textsf{ says } A'$

1. $\cdot \vdash_G (\Box_{K'}(K' \supset \ulcorner A' \urcorner)) \supset \Box_{K'} \Box_{K'} (K' \supset \ulcorner A' \urcorner)$ (Rule (ax) and 4)
2. $\Box_{K'}(K' \supset \ulcorner A' \urcorner) \vdash_G \Box_{K'} \Box_{K'} (K' \supset \ulcorner A' \urcorner)$ (Theorem E.2)
3. $\cdot \vdash_G (\Box_{K'}(K' \supset \ulcorner A' \urcorner)) \supset (K' \supset (\Box_{K'}(K' \supset \ulcorner A' \urcorner)))$ (Rule (ax) and (imp1))
4. $\cdot \vdash_G \Box_{K'}((\Box_{K'}(K' \supset \ulcorner A' \urcorner)) \supset (K' \supset (\Box_{K'}(K' \supset \ulcorner A' \urcorner))))$ (Rule (nec))

5. $\cdot \vdash_G (\Box_{K'} \Box_{K'} (K' \supset \ulcorner A' \urcorner)) \supset \Box_{K'}(K' \supset (\Box_{K'}(K' \supset \ulcorner A' \urcorner)))$ (Rule (ax), (K) and rule (mp))
6. $\Box_{K'} \Box_{K'} (K' \supset \ulcorner A' \urcorner) \vdash_G \Box_{K'}(K' \supset (\Box_{K'}(K' \supset \ulcorner A' \urcorner)))$ (Theorem E.2)
7. $\Box_{K'}(K' \supset \ulcorner A' \urcorner) \vdash_G \Box_{K'}(K' \supset (\Box_{K'}(K' \supset \ulcorner A' \urcorner)))$ (Lemma E.1.2 on 2 and 6)
8. $\cdot \vdash_G (\Box_{K'}(K' \supset \ulcorner A' \urcorner)) \supset \Box_{K'}(K' \supset (\Box_{K'}(K' \supset \ulcorner A' \urcorner)))$ (Theorem E.2)
9. $\mathcal{O} \vdash_G (\Box_{K'}(K' \supset \ulcorner A' \urcorner)) \supset \Box_{K'}(K' \supset (\Box_{K'}(K' \supset \ulcorner A' \urcorner)))$ (Weakening, Lemma E.1.1)

Case. (Axiom C) $A = K' \mathsf{\ says\ } ((K' \mathsf{\ says\ } A') \supset A')$

1. $\Box_{K'}(K' \supset \ulcorner A' \urcorner), K' \vdash_G (\Box_{K'}(K' \supset \ulcorner A' \urcorner)) \supset (K' \supset \ulcorner A' \urcorner)$ (Rule (ax) and (T))
2. $\Box_{K'}(K' \supset \ulcorner A' \urcorner), K' \vdash_G \Box_{K'}(K' \supset \ulcorner A' \urcorner)$ (Rule (use))
3. $\Box_{K'}(K' \supset \ulcorner A' \urcorner), K' \vdash_G K' \supset \ulcorner A' \urcorner$ (Rule (mp))
4. $\Box_{K'}(K' \supset \ulcorner A' \urcorner), K' \vdash_G K'$ (Rule (use))
5. $\Box_{K'}(K' \supset \ulcorner A' \urcorner), K' \vdash_G \ulcorner A' \urcorner$ (Rule (mp))
6. $\cdot \vdash_G K' \supset ((\Box_{K'}(K' \supset \ulcorner A' \urcorner)) \supset \ulcorner A' \urcorner)$ (Theorem E.2)
7. $\cdot \vdash_G \Box_{K'}(K' \supset ((\Box_{K'}(K' \supset \ulcorner A' \urcorner)) \supset \ulcorner A' \urcorner))$ (Rule (nec))
8. $\mathcal{O} \vdash_G \Box_{K'}(K' \supset ((\Box_{K'}(K' \supset \ulcorner A' \urcorner)) \supset \ulcorner A' \urcorner))$ (Weakening, Lemma E.1.1)

Case. (Axiom S) $A = (K'_1 \mathsf{\ says\ } A') \supset (K'_2 \mathsf{\ says\ } A')$

1. $\cdot \vdash_G (K'_2 \supset K'_1) \supset ((K'_1 \supset \ulcorner A' \urcorner) \supset (K'_2 \supset \ulcorner A' \urcorner))$ (Basic propositional theorem)
2. $\cdot \vdash_G \Box_{K'_2}((K'_2 \supset K'_1) \supset ((K'_1 \supset \ulcorner A' \urcorner) \supset (K'_2 \supset \ulcorner A' \urcorner)))$ (Rule (nec))
3. $\cdot \vdash_G (\Box_{K'_2}(K'_2 \supset K'_1)) \supset ((\Box_{K'_2}(K'_1 \supset \ulcorner A' \urcorner)) \supset \Box_{K'_2}(K'_2 \supset \ulcorner A' \urcorner))$ (Rule (ax), (K), and rule (mp))
4. $\Box_{K'_2}(K'_2 \supset K'_1), \Box_{K'_2}(K'_1 \supset \ulcorner A' \urcorner) \vdash_G \Box_{K'_2}(K'_2 \supset \ulcorner A' \urcorner)$ (Theorem E.2)
5. $\cdot \vdash_G (\Box_{\ell}(K'_2 \supset K'_1)) \supset \Box_{K'_2}(K'_2 \supset K'_1)$ (Rule (ax) and (S))
6. $\Box_{\ell}(K'_2 \supset K'_1) \vdash_G \Box_{K'_2}(K'_2 \supset K'_1)$ (Theorem E.2)
7. $\cdot \vdash_G (\Box_{K'_1}(K'_1 \supset \ulcorner A' \urcorner)) \supset \Box_{K'_2}(K'_1 \supset \ulcorner A' \urcorner)$ (Rule (ax) and (S))
8. $\Box_{K'_1}(K'_1 \supset \ulcorner A' \urcorner) \vdash_G \Box_{K'_2}(K'_1 \supset \ulcorner A' \urcorner)$ (Theorem E.2)
9. $\Box_{\ell}(K'_2 \supset K'_1), \Box_{K'_2}(K'_1 \supset \ulcorner A' \urcorner) \vdash_G \Box_{K'_2}(K'_2 \supset \ulcorner A' \urcorner)$ (Lemma E.1.2 on 6 and 4)
10. $\Box_{\ell}(K'_2 \supset K'_1), \Box_{K'_1}(K'_1 \supset \ulcorner A' \urcorner) \vdash_G \Box_{K'_2}(K'_2 \supset \ulcorner A' \urcorner)$ (Lemma E.1.2 on 8 and 9)
11. $\Box_{\ell}(K'_2 \supset K'_1) \vdash_G (\Box_{K'_1}(K'_1 \supset \ulcorner A' \urcorner)) \supset \Box_{K'_2}(K'_2 \supset \ulcorner A' \urcorner)$ (Theorem E.2)
12. $\mathcal{O} \vdash_G (\Box_{K'_1}(K'_1 \supset \ulcorner A' \urcorner)) \supset \Box_{K'_2}(K'_2 \supset \ulcorner A' \urcorner)$ (Weakening, Lemma E.1.1)

The remaining cases are straightforward.

$\Box$

E.3 Proof of Completeness

Our proof of completeness of the translation from DTL_0 to $\mathrm{CS4}^m$ is semantic, and uses Kripke models of DTL_0 described in Section 4 and Appendix D. At a high level, the steps in the proof are the following. First we define an interpretation of the formulas of $\mathrm{CS4}^m$ in Kripke models of DTL_0, and show that the interpretation is sound. This is rather unusual, and works because the logics DTL_0 and $\mathrm{CS4}^m$ are quite similar. Next we show that for any DTL_0 formula A, it is the case that $\models \ulcorner A \urcorner$ in this interpretation if and only if $\models A$ in the usual Kripke interpretation of DTL_0. Then, completeness of the translation follows from completeness of DTL_0 with respect to its Kripke models (Theorem D.7).

Definition E.5 (Kripke Interpretation of $\mathrm{CS4}^m$)**.** Let $(W, \theta, \leq, (\sqsubseteq_K)^{K \in \mathtt{Prin}}, \rho, F)$ be a Kripke model for DTL_0. Then for $\mathrm{CS4}^m$ formulas, we define satisfaction at a world w by induction on formulas as follows:

$w \models P$ iff $P \in \rho(w)$.

$w \models K$ iff $w \in F$ or $K \in \theta(w)$.

$w \models A \wedge B$ iff $w \models A$ and $w \models B$.

$w \models A \vee B$ iff $w \models A$ or $w \models B$.

$w \models \top$.

$w \models \perp$ iff $w \in F$.

$w \models A \supset B$ iff for all w', $w \leq w'$ and $w' \models A$ imply $w' \models B$.

$w \models \Box_K A$ iff either $w \in F$ or (for all w', w'', $w \leq w' \sqsubseteq_K w''$ implies $w'' \models A$, and for all w', $w \leq w'$ implies $w' \models A$).

Lemma E.6 (Monotonicity)**.** *For any CS4^m formula A, $w \models A$ and $w \leq w'$ imply $w' \models A$.*

Proof. By induction on A. Most cases work as for the proof of Lemma D.1. The only new cases here are $A = K$ and $A = \Box_K A'$.

Case. $A = K$. Since $w \models K$, $w \in F$ or $K \in \theta(w)$. In the former case, by condition (F-her), $w' \in F$. Thus $w' \models K$. In the latter case, by condition (Imp-mon), $K \in \theta(w')$. Thus $w' \models K$.

Case. $A = \Box_K A'$. We need to show that $w' \models \Box_K A'$. We assume that $w \notin F$, because otherwise $w' \in F$ by (F-her), and trivially we would have $w' \models \Box_K A'$. Pick any w_2, w_3 such that $w' \leq w_2 \sqsubseteq_K w_3$. Clearly, $w \leq w_2 \sqsubseteq_K w_3$. Hence by assumptions $w \models \Box_K A'$ and $w \notin F$, we get $w_3 \models A'$. Next, pick any $w'' \geq w'$. Clearly, $w \leq w''$. Hence by assumptions $w \models \Box_K A'$ and $w \notin F$, we get $w'' \models A'$. Thus $w' \models \Box_K A'$.

$\square$

Lemma E.7 (Falsehood)**.** *For any CS4^m formula A, $w \models \perp$ implies $w \models A$.*

Proof. Exactly like that of lemma D.2. □

As for DTL_0, given a Kripke model M we say that $M \models A$ if for each $w \in M$, $w \models A$. This interpretation is sound in the following sense.

Lemma E.8 (Soundness of Interpretation)**.** *If* $\vdash A$ *in* $CS4^m$*, then for each* DTL_0 *Kripke model* M*,* $M \models A$*.*

Proof. Pick any model M. We induct on the derivation of $\vdash A$ in $\mathrm{CS4}^m$ to show that for each $w \in M$, $w \models A$. We case analyze the last rule in the derivation of $\vdash A$.

Case. $\dfrac{\vdash A}{\vdash \Box_K A}\text{nec}$

Pick any w', w'' such that $w \leq w' \sqsubseteq_K w''$. By i.h., $w'' \models A$. Next, pick any w' such that $w \leq w'$. By i.h., $w' \models A$. Thus by definition of satisfaction, $w \models \Box_K A$.

Case. $\dfrac{\vdash A \supset B \qquad \vdash A}{\vdash B}\text{mp}$

1. $w \models A \supset B$ (i.h. on 1st premise)
2. $w \models A$ (i.h. on 2nd premise)
3. $w \leq w$ (Reflexivity of $\leq$)
4. $w \models A$ implies $w \models B$ (Defn. of satisfaction, 1, 3)
5. $w \models B$ (2, 4)

Case. $\dfrac{A \text{ is an axiom}}{\vdash A}\text{ax}$

We analyze the possible axioms A.

Case. (Axiom K) $A = (\Box_{K'}(A' \supset B')) \supset ((\Box_{K'}A') \supset (\Box_{K'}B'))$

Pick any $w' \geq w$, and assume that $w' \models \Box_{K'}(A' \supset B')$. It suffices to show that $w' \models (\Box_{K'}A') \supset (\Box_{K'}B')$. Now pick any $w'' \geq w'$ and assume that $w'' \models \Box_{K'}A'$. Then it suffices to show that $w'' \models \Box_{K'}B'$. We may assume that $w, w', w'' \notin F$ because otherwise, by condition (F-her), $w'' \in F$, and trivially we would have $w'' \models \Box_{K'}B'$.

Pick w_3, w_4 such that $w'' \leq w_3 \sqsubseteq_{K'} w_4$. Observe that $w' \leq w_3 \sqsubseteq_{K'} w_4$. From assumptions $w' \models \Box_{K'}(A' \supset B')$ and $w' \notin F$ it follows that $w_4 \models A' \supset B'$. Similarly, from assumptions $w'' \models \Box_{K'}A'$ and $w'' \notin F$ it follows that $w_4 \models A'$. Clearly, then $w_4 \models B'$.

Next pick w_3 such that $w'' \leq w_3$. Observe that $w' \leq w_3$. From assumptions $w' \models \Box_{K'}(A' \supset B')$ and $w' \notin F$ it follows that $w_3 \models A' \supset B'$. Similarly, from assumptions $w'' \models \Box_{K'}A'$ and $w'' \notin F$ it follows that $w_3 \models A'$. Thus $w_3 \models B'$.

It follows from the definition of satisfaction for $\Box_{K'}B'$ that $w'' \models \Box_{K'}B'$.

Case. (Axiom 4) $A = (\Box_{K'} A') \supset \Box_{K'} \Box_{K'} A'$

Pick any $w' \geq w$, and assume that $w' \models \Box_{K'} A'$. It suffices to show that $w' \models \Box_{K'} \Box_{K'} A'$. We may assume that $w' \notin F$ because otherwise $w' \models \Box_{K'} \Box_{K'} A'$ trivially by definition of satisfaction.

Pick any w_2, w_3 such that $w' \leq w_2 \sqsubseteq_{K'} w_3$. We must show that $w_3 \models \Box_{K'} A'$. So pick any w_4, w_5 such that $w_3 \leq w_4 \sqsubseteq_{K'} w_5$. We must show that $w_5 \models A'$. Observe that $w' \leq w_2 \sqsubseteq_{K'} w_3 \leq w_4 \sqsubseteq_{K'} w_5$. By (commutativity), $w' \leq w_2 \sqsubseteq_{K'} w_4 \sqsubseteq_{K'} w_5$. By (mod-trans), $w' \leq w_2 \sqsubseteq_{K'} w_5$. It follows from assumptions $w' \models \Box_{K'} A'$ and $w' \notin F$ that $w_5 \models A'$. Next, pick any w_4 such that $w_3 \leq w_4$. We must show that $w_4 \models A'$. Observe that $w' \leq w_2 \sqsubseteq_{K'} w_3 \leq w_4$. By (commutativity), $w' \leq w_2 \sqsubseteq_{K'} w_4$. It follows from assumptions $w' \models \Box_{K'} A'$ and $w' \notin F$ that $w_4 \models A'$. Hence $w_3 \models \Box_{K'} A$.

Next pick w_2 such that $w' \leq w_2$. We must show that $w_2 \models \Box_{K'} A'$. So pick any w_3, w_4 such that $w_2 \leq w_3 \sqsubseteq_{K'} w_4$. We must show that $w_4 \models A'$. Observe that $w' \leq w_2 \sqsubseteq_{K'} w_4$. It follows from assumptions $w' \models \Box_{K'} A'$ and $w' \notin F$ that $w_4 \models A'$. Finally pick w_3 such that $w_2 \leq w_3$. We must show that $w_3 \models A'$. Observe that $w' \leq w_3$. By assumptions $w' \models \Box_{K'} A'$ and $w' \notin F$ it follows that $w_3 \models A'$. Hence $w_2 \models \Box_{K'} A'$.

Thus $w' \models \Box_{K'} \Box_{K'} A'$.

Case. (Axiom T) $A = (\Box_{K'} A') \supset A'$

Pick any $w' \geq w$, and assume that $w' \models \Box_{K'} A'$. It suffices to show that $w' \models A$. We may assume that $w' \notin F$, else $w' \models \bot$ and by Lemma E.7, $w' \models A'$. Now observe that $w' \leq w'$. Hence by definition of satisfaction of $\Box_{K'} A'$, we must have $w' \models A'$ as required.

Case. (Axiom S) $A = (\Box_K A') \supset \Box_{K'} A'$, where $K \succeq K'$

Pick any $w' \geq w$, and assume that $w' \models \Box_K A'$. It suffices to show that $w' \models \Box_{K'} A'$. We may assume that $w' \notin F$, else we trivially have $w' \models \Box_{K'} A'$ by definition of satisfaction.

Pick any w_2, w_3 such that $w' \leq w_2 \sqsubseteq_{K'} w_3$. We must show that $w_3 \models A'$. By (mod-closure), $w' \leq w_2 \sqsubseteq_K w_3$. It follows by assumptions $w' \models \Box_K A'$ and $w' \notin F$ that $w_3 \models A'$.

Next pick any w_2 such that $w' \leq w_2$. We must show that $w_2 \models A'$. This follows immediately by assumptions $w' \models \Box_K A'$ and $w' \notin F$.

Thus $w' \models \Box_{K'} A'$, as required.

The remaining cases are straightforward, as they do not rely on modalities.

$\Box$

Now we prove a critical lemma, which states that $w \models \ulcorner A \urcorner$ if and only if $w \models A$, for each DTL$_0$ formula A.

Lemma E.9 (Critical Lemma). *For each DTL_0 formula A, each Kripke model M, and each $w \in M$, it is the case that $w \models A$ if and only if $w \models \ulcorner A \urcorner$.*

Proof. We induct on A, and analyze cases on the top constructor in it.

Case. $A = P$ (A is atomic). $\ulcorner A \urcorner = P$.
By definition, $w \models A$ iff $w \in \rho(P)$ iff $w \models \ulcorner A \urcorner$.

Case. $A = A_1 \wedge A_2$. $\ulcorner A \urcorner = \ulcorner A_1 \urcorner \wedge \ulcorner A_2 \urcorner$.

$$\begin{array}{lll} w \models A_1 \wedge A_2 & \text{iff} \quad w \models A_1 \text{ and } w \models A_2 & \text{(Defn.)} \\ & \text{iff} \quad w \models \ulcorner A_1 \urcorner \text{ and } w \models \ulcorner A_2 \urcorner & \text{(i.h.)} \\ & \text{iff} \quad w \models \ulcorner A_1 \urcorner \wedge \ulcorner A_2 \urcorner & \text{(Defn.)} \end{array}$$

Case. $A = A_1 \vee A_2$. $\ulcorner A \urcorner = \ulcorner A_1 \urcorner \vee \ulcorner A_2 \urcorner$.

$$\begin{array}{lll} w \models A_1 \vee A_2 & \text{iff} \quad w \models A_1 \text{ or } w \models A_2 & \text{(Defn.)} \\ & \text{iff} \quad w \models \ulcorner A_1 \urcorner \text{ or } w \models \ulcorner A_2 \urcorner & \text{(i.h.)} \\ & \text{iff} \quad w \models \ulcorner A_1 \urcorner \vee \ulcorner A_2 \urcorner & \text{(Defn.)} \end{array}$$

Case. $A = A_1 \supset A_2$. $\ulcorner A \urcorner = \ulcorner A_1 \urcorner \supset \ulcorner A_2 \urcorner$.
Suppose $w \models A_1 \supset A_2$. Pick any $w' \geq w$ and assume that $w' \models \ulcorner A_1 \urcorner$. It suffices to show that $w' \models \ulcorner A_2 \urcorner$. By i.h., $w' \models A_1$. By assumption $w \models A_1 \supset A_2$, $w' \models A_1$ implies $w' \models A_2$. Thus $w' \models A_2$. By i.h., $w' \models \ulcorner A_2 \urcorner$ as required.

Conversely, suppose $w \models \ulcorner A_1 \urcorner \supset \ulcorner A_2 \urcorner$. Pick any $w' \geq w$ and assume that $w' \models A_1$. It suffices to show that $w' \models A_2$. By i.h., $w' \models \ulcorner A_1 \urcorner$. By assumption $w \models \ulcorner A_1 \urcorner \supset \ulcorner A_2 \urcorner$, $w' \models \ulcorner A_1 \urcorner$ implies $w' \models \ulcorner A_2 \urcorner$. Thus $w' \models \ulcorner A_2 \urcorner$. By i.h., $w' \models A_2$ as required.

Case. $A = \top$. $\ulcorner A \urcorner = \top$.
This case is trivial because $w \models \top$ for each w.

Case. $A = \bot$. $\ulcorner A \urcorner = \bot$.
By definition, $w \models A$ iff $w \in F$ iff $w \models \ulcorner A \urcorner$.

Case. $A = K \textsf{ says } B$. $\ulcorner A \urcorner = \Box_K(K \supset \ulcorner B \urcorner)$.
Suppose $w \models K \textsf{ says } B$. We must show that $w \models \Box_K(K \supset \ulcorner B \urcorner)$. We may assume that $w \notin F$, otherwise by definition we have $w \models \Box_K(K \supset \ulcorner B \urcorner)$.

Pick any w_1, w_2 such that $w \leq w_1 \sqsubseteq_K w_2$. We need to show that $w_2 \models K \supset \ulcorner B \urcorner$. So pick any w_3 such that $w_2 \leq w_3$ and $w_3 \models K$. It suffices to show that $w_3 \models \ulcorner B \urcorner$. Now observe that $w \leq w_1 \sqsubseteq_K w_2 \leq w_3$. By (commutativity), $w \leq w_1 \sqsubseteq_K w_3$. By assumptions $w \models K \textsf{ says } B$ and $w \notin F$, we get $w_3 \models B$. By i.h., $w_3 \models \ulcorner B \urcorner$, as required.

Next, pick any w_1 such that $w \leq w_1$. We need to show that $w_1 \models K \supset \ulcorner B \urcorner$. Pick any w_2 such that $w_1 \leq w_2$ and $w_2 \models K$. It suffices to show that $w_2 \models \ulcorner B \urcorner$. From assumption $w_2 \models K$, we get $w_2 \in F$ or $K \in \theta(w_2)$. If $w_2 \in F$, then by Lemma E.7, $w_2 \models \ulcorner B \urcorner$. If $K \in \theta(w_2)$, by (mod-refl) we get $w_2 \sqsubseteq_K w_2$. We therefore have $w \leq w_2 \sqsubseteq_K w_2$. By assumptions $w \models K \textsf{ says } B$ and $w \notin F$, we get $w_2 \models B$. By i.h., $w_2 \models \ulcorner B \urcorner$, as required. Thus $w \models \Box_K(K \supset \ulcorner B \urcorner)$.

Conversely, suppose that $w \models \Box_K(K \supset \ulcorner B \urcorner)$. We must show that $w \models K \textsf{ says } B$. We may assume that $w \notin F$, otherwise by definition we have $w \models K \textsf{ says } B$.

Pick any w_1, w_2 such that $w \leq w_1 \sqsubseteq_K w_2$. It suffices to show that $w_2 \models B$. By assumptions $w \models \Box_K(K \supset \ulcorner B \urcorner)$ and $w \notin F$ it follows that $w_2 \models K \supset \ulcorner B \urcorner$. Hence, $w_2 \models K$ implies $w_2 \models \ulcorner B \urcorner$. Now, by definition of $\sqsubseteq_K$, $K \in \theta(w_2)$. Thus, $w_2 \models K$. This gives us $w_2 \models \ulcorner B \urcorner$. By i.h., $w_2 \models B$ as required. $\square$

We need one last lemma before we establish soundness and completeness. This lemma states that $\mathcal{O}$ is satisfied in all Kripke models.

Lemma E.10 (Satisfaction for orders)**.** *For every Kripke model M, and every $w \in M$, $w \models \mathcal{O}$.*

Proof. We show that $w \models \Box_\ell(K_1 \supset K_2)$ whenever $K_2 \succeq K_1$. Pick any w', w'' such that $w \leq w' \sqsubseteq_\ell w''$. We need to show that $w'' \models K_1 \supset K_2$. Pick any $w_3 \geq w''$ and assume that $w_3 \models K_1$. It suffices to show that $w_3 \models K_2$. By assumption $w_3 \models K_1$, we get $w_3 \in F$ or $K_1 \in \theta(w_3)$. If $w_3 \in F$, then $w_3 \models K_2$ by definition. If $K_1 \in \theta(w_3)$, then by (view-closure) and $K_2 \succeq K_1$, we get $K_2 \in \theta(w_3)$. Thus, $w_3 \models K_2$ as required.

Next pick w' such that $w' \geq w$. We need to show that $w' \models K_1 \supset K_2$. Pick any $w_2 \geq w'$ and assume that $w_2 \models K_1$. It suffices to show that $w_2 \models K_2$. By assumption $w_2 \models K_1$, we get $w_2 \in F$ or $K_1 \in \theta(w_2)$. If $w_2 \in F$, then $w_2 \models K_2$ by definition. If $K_1 \in \theta(w_2)$, then by (view-closure) and $K_2 \succeq K_1$, we get $K_2 \in \theta(w_2)$. Thus, $w_2 \models K_2$ as required.

Hence $w \models \Box_\ell(K_1 \supset K_2)$ whenever $K_2 \succeq K_1$, and consequently, $w \models \mathcal{O}$. $\square$

Theorem E.11 (Correctness; Theorem 5.2)**.** *$\cdot \xrightarrow{K} A$ in DTL$_0$ if and only if $\vdash \mathcal{O} \supset (K \supset \ulcorner A \urcorner)$ in CS4^m.*

Proof. Suppose $\cdot \xrightarrow{K} A$. By Corollary C.10, $\vdash_H K$ says A in DTL$_0$'s axiomatic system. By Lemma E.4, $\mathcal{O} \vdash_G \Box_K(K \supset \ulcorner A \urcorner)$ in CS4^m's axiomatic system. Since $\mathcal{O} \vdash_G (\Box_K(K \supset \ulcorner A \urcorner)) \supset (K \supset \ulcorner A \urcorner)$ by Axiom (T), we get $\mathcal{O} \vdash_G K \supset \ulcorner A \urcorner$ by (mp). By Theorem E.2, $\cdot \vdash_G \mathcal{O} \supset (K \supset \ulcorner A \urcorner)$, and by Theorem E.3, $\vdash \mathcal{O} \supset (K \supset \ulcorner A \urcorner)$.

Conversely, suppose that $\vdash \mathcal{O} \supset (K \supset \ulcorner A \urcorner)$ in CS4^m. By Lemma E.8, for each Kripke model M, and each world w, $w \models \mathcal{O} \supset (K \supset \ulcorner A \urcorner)$. From Lemma E.10, $w \models \mathcal{O}$. Therefore, $w \models K \supset \ulcorner A \urcorner$. Since w is arbitrary, $M \models K \supset \ulcorner A \urcorner$. Now pick any $w \in W^K$. By definition, $w \models K$. Using this fact and $M \models K \supset \ulcorner A \urcorner$, we get $w \models \ulcorner A \urcorner$. Using Lemma E.9, we deduce $w \models A$. Since w is arbitrarily chosen in W^K, we get $M \models^K A$. Since M is arbitrary, by Theorem D.7 we obtain $\cdot \xrightarrow{K} A$. $\square$

F Proofs from Section 5.2

In this Appendix we prove that the translation from ICL to DTL$_0$ is both sound and complete (Theorem 5.3). We use a sequent calculus formulation of ICL that is shown in Figure 6.[10] This sequent calculus is taken from earlier work [33] (For a slightly more detailed, tutorial explanation see [30]). It uses two categorical judgments: A true and K affirms A. The latter means that principal K states that A is true. Just as K says A

[10]We could have avoided using the sequent calculus for ICL, and worked with the axiomatic system, without any significant change to the proof method. Using the sequent calculus eliminates trivial steps that often arise in formal axiomatic proofs, and compacts our proofs.

$$\frac{(P \text{ atomic})}{\Gamma, P \vdash P}\text{init} \qquad \frac{\Gamma \vdash A}{\Gamma \vdash K\ \mathsf{affirms}\ A}\text{affs}$$

$$\frac{\Gamma \vdash K\ \mathsf{affirms}\ A}{\Gamma \vdash K\ \mathsf{says}\ A}\text{saysR} \qquad \frac{\Gamma, K\ \mathsf{says}\ A, A \vdash K\ \mathsf{affirms}\ C}{\Gamma, K\ \mathsf{says}\ A \vdash K\ \mathsf{affirms}\ C}\text{saysL}$$

$$\frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \wedge B}\wedge\text{R} \qquad \frac{\Gamma, A \wedge B, A, B \vdash C}{\Gamma, A \wedge B \vdash C}\wedge\text{L} \qquad \frac{\Gamma, A \wedge B, A, B \vdash K\ \mathsf{affirms}\ C}{\Gamma, A \wedge B \vdash K\ \mathsf{affirms}\ C}\wedge\text{Laff}$$

$$\frac{\Gamma \vdash A}{\Gamma \vdash A \vee B}\vee\text{R}_1 \qquad \frac{\Gamma \vdash B}{\Gamma \vdash A \vee B}\vee\text{R}_2 \qquad \frac{\Gamma, A \vee B, A \vdash C \qquad \Gamma, A \vee B, B \vdash C}{\Gamma, A \vee B \vdash C}\vee\text{L}$$

$$\frac{\Gamma, A \vee B, A \vdash K\ \mathsf{affirms}\ C \qquad \Gamma, A \vee B, B \vdash K\ \mathsf{affirms}\ C}{\Gamma, A \vee B \vdash K\ \mathsf{affirms}\ C}\vee\text{Laff}$$

$$\frac{}{\Gamma \vdash \top}\top\text{R} \qquad \frac{}{\Gamma, \bot \vdash C}\bot\text{L} \qquad \frac{}{\Gamma, \bot \vdash K\ \mathsf{affirms}\ C}\bot\text{Laff}$$

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \supset B}\supset\text{R} \qquad \frac{\Gamma, A \supset B \vdash A \qquad \Gamma, A \supset B, B \vdash C}{\Gamma, A \supset B \vdash C}\supset\text{L}$$

$$\frac{\Gamma, A \supset B \vdash A \qquad \Gamma, A \supset B, B \vdash K\ \mathsf{affirms}\ C}{\Gamma, A \supset B \vdash K\ \mathsf{affirms}\ C}\supset\text{Laff}$$

Figure 6: Sequent Calculus for ICL

internalizes $K\ \mathsf{claims}\ A$ in DTL_0, $K\ \mathsf{says}\ A$ internalizes $K\ \mathsf{affirms}\ A$ in ICL. Unlike DTL_0 where the judgment $K\ \mathsf{claims}\ A$ only occurs in the hypothesis, but never as the conclusion of a sequent, the judgment $K\ \mathsf{affirms}\ A$ in ICL occurs only in the conclusions of sequents and never in the hypothesis. This shift is due to the difference in the nature of says in the two logics. As usual, the judgment name true is often elided. It is quite easy to show that this sequent calculus is equivalent to the axiomatic presentation for ICL (Section 5.2).

Lemma F.1. *$\vdash A$ in ICL's axiomatic system if and only if $\cdot \vdash A$ in ICL's sequent calculus.*

Proof. It is easy to show by induction that $\vdash A$ implies $\cdot \vdash A$. For the proof in the other direction, we generalize the statement, and prove by induction that whenever $\Gamma \vdash A$ in the sequent calculus, then $\vdash \Gamma \supset A$ in the axiomatic system. (This requires that we establish the deduction theorem for the axiomatic system, but that is straighforward.) □

The sequent calculus for ICL admits two cut principles, as the following lemma states.

Lemma F.2 (Admissibility of Cut)**.** *The following hold for ICL's sequent calculus.*

1. $\Gamma \vdash A$ *and* $\Gamma, A \vdash C$ *imply* $\Gamma \vdash C$

2. $\Gamma \vdash K \text{ affirms } A$ *and* $\Gamma, A \vdash K \text{ affirms } C$ *imply* $\Gamma \vdash K \text{ affirms } C$.

Proof. See [33], Theorem 2, clauses 4 and 5. □

Similarly, the identity property also holds for ICL's sequent calculus.

Lemma F.3 (Identity)**.** *For each formula* A*, it is the case that* $\Gamma, A \vdash A$.

Proof. See [33], Theorem 2, clause 2. □

F.1 Proof of Soundness

Before proving soundness we prove a basic lemma.

Lemma F.4 (Global Lemma)**.** *For each ICL formula* A *and each principal* K *in* DTL_0*, it is the case that* $\ulcorner A \urcorner \xrightarrow{K} \text{global } \ulcorner A \urcorner$ *in* DTL_0.

Proof. We induct on A, and case analyze its top constructor.

Case. A is atomic.

1. $\ell \text{ claims } A, A \xrightarrow{\ell} A$ (Rule (init))
2. $\ell \text{ claims } A \xrightarrow{\ell} A$ (Rule (claims))
3. $\ell \text{ claims } A \xrightarrow{\ell} \ell \text{ says } A$ (Rule (saysR))
4. $\ell \text{ claims } A, \ell \text{ says } A \xrightarrow{K} \ell \text{ says } \ell \text{ says } A$ (Rule (saysR))
5. $\ell \text{ says } A \xrightarrow{K} \ell \text{ says } \ell \text{ says } A$ (Rule (saysL))

Case. $A = A_1 \wedge A_2$

1. $\ell \text{ claims } \ulcorner A \urcorner, \ell \text{ claims } \ulcorner B \urcorner, \ulcorner A \urcorner \xrightarrow{\ell} \ulcorner A \urcorner$ (Theorem B.3)
2. $\ell \text{ claims } \ulcorner A \urcorner, \ell \text{ claims } \ulcorner B \urcorner \xrightarrow{\ell} \ulcorner A \urcorner$ (Rule (claims))
3. $\ell \text{ claims } \ulcorner A \urcorner, \ell \text{ claims } \ulcorner B \urcorner, \ulcorner B \urcorner \xrightarrow{\ell} \ulcorner B \urcorner$ (Theorem B.3)
4. $\ell \text{ claims } \ulcorner A \urcorner, \ell \text{ claims } \ulcorner B \urcorner \xrightarrow{\ell} \ulcorner B \urcorner$ (Rule (claims))
5. $\ell \text{ claims } \ulcorner A \urcorner, \ell \text{ claims } \ulcorner B \urcorner \xrightarrow{\ell} \ulcorner A \urcorner \wedge \ulcorner B \urcorner$ (Rule ($\wedge$R) on 2 an 4)
6. $\ell \text{ claims } \ulcorner A \urcorner, \ell \text{ claims } \ulcorner B \urcorner, \ell \text{ says } \ulcorner A \urcorner, \ell \text{ says } \ulcorner B \urcorner, \ulcorner A \urcorner \wedge \ulcorner B \urcorner \xrightarrow{K} \ell \text{ says } (\ulcorner A \urcorner \wedge \ulcorner B \urcorner)$ (Rule (saysR))
7. $\ell \text{ says } \ulcorner A \urcorner, \ell \text{ says } \ulcorner B \urcorner, \ulcorner A \urcorner \wedge \ulcorner B \urcorner \xrightarrow{K} \ell \text{ says } (\ulcorner A \urcorner \wedge \ulcorner B \urcorner)$ (Rule (saysL) twice)

8. $\ulcorner A \urcorner \xrightarrow{K} \ell \text{ says } \ulcorner A \urcorner$ (i.h. on A)

9. $\ulcorner A \urcorner, \ell \text{ says } \ulcorner B \urcorner, \ulcorner A \urcorner \wedge \ulcorner B \urcorner \xrightarrow{K} \ell \text{ says } (\ulcorner A \urcorner \wedge \ulcorner B \urcorner)$ (Theorem B.2 on 8 and 7)

10. $\ulcorner B \urcorner \xrightarrow{K} \ell \text{ says } \ulcorner B \urcorner$ (i.h. on B)

11. $\ulcorner A \urcorner, \ulcorner B \urcorner, \ulcorner A \urcorner \wedge \ulcorner B \urcorner \xrightarrow{K} \ell \text{ says } (\ulcorner A \urcorner \wedge \ulcorner B \urcorner)$ (Theorem B.2 on 10 and 9)

12. $\ulcorner A \urcorner \wedge \ulcorner B \urcorner \xrightarrow{K} \ell \text{ says } (\ulcorner A \urcorner \wedge \ulcorner B \urcorner)$ (Rule ($\wedge$L))

Case. $A = A_1 \vee A_2$

1. $\ell \text{ claims } \ulcorner A \urcorner, \ulcorner A \urcorner \xrightarrow{\ell} \ulcorner A \urcorner$ (Theorem B.3)
2. $\ell \text{ claims } \ulcorner A \urcorner, \ulcorner A \urcorner \xrightarrow{\ell} \ulcorner A \urcorner \vee \ulcorner B \urcorner$ (Rule ($\vee$ R$_1$))
3. $\ell \text{ claims } \ulcorner A \urcorner \xrightarrow{\ell} \ulcorner A \urcorner \vee \ulcorner B \urcorner$ (Rule (claims))
4. $\ell \text{ claims } \ulcorner A \urcorner, \ell \text{ says } \ulcorner A \urcorner, \ulcorner A \urcorner \vee \ulcorner B \urcorner \xrightarrow{K} \ell \text{ says } (\ulcorner A \urcorner \vee \ulcorner B \urcorner)$ (Rule (saysR))
5. $\ell \text{ says } \ulcorner A \urcorner, \ulcorner A \urcorner \vee \ulcorner B \urcorner \xrightarrow{K} \ell \text{ says } (\ulcorner A \urcorner \vee \ulcorner B \urcorner)$ (Rule (saysL))
6. $\ulcorner A \urcorner \xrightarrow{K} \ell \text{ says } \ulcorner A \urcorner$ (i.h. on A)
7. $\ulcorner A \urcorner, \ulcorner A \urcorner \vee \ulcorner B \urcorner \xrightarrow{K} \ell \text{ says } (\ulcorner A \urcorner \vee \ulcorner B \urcorner)$ (Theorem B.2 on 6 and 5)
8. $\ell \text{ says } \ulcorner B \urcorner, \ulcorner A \urcorner \vee \ulcorner B \urcorner \xrightarrow{K} \ell \text{ says } (\ulcorner A \urcorner \vee \ulcorner B \urcorner)$ (Similar to 5)
9. $\ulcorner B \urcorner \xrightarrow{K} \ell \text{ says } \ulcorner B \urcorner$ (i.h. on B)
10. $\ulcorner B \urcorner, \ulcorner A \urcorner \vee \ulcorner B \urcorner \xrightarrow{K} \ell \text{ says } (\ulcorner A \urcorner \vee \ulcorner B \urcorner)$ (Theorem B.2 on 9 and 8)
11. $\ulcorner A \urcorner \vee \ulcorner B \urcorner \xrightarrow{K} \ell \text{ says } (\ulcorner A \urcorner \vee \ulcorner B \urcorner)$ (Rule ($\vee$L) on 7 and 10)

Case. $A = \top$

1. $\cdot \xrightarrow{\ell} \top$ (Rule ($\top$R))
2. $\top \xrightarrow{K} \ell \text{ says } \top$ (Rule (saysR))

Case. $A = \bot$

1. $\bot \xrightarrow{K} \ell \text{ says } \bot$ (Rule ($\bot$L))

Case. $A = A_1 \supset A_2$

1. $\ell \text{ claims } \ulcorner A_1 \urcorner \supset \ulcorner A_2 \urcorner, \ulcorner A_1 \urcorner \supset \ulcorner A_2 \urcorner \xrightarrow{\ell} \ulcorner A_1 \urcorner \supset \ulcorner A_2 \urcorner$ (Theorem B.3)
2. $\ell \text{ claims } \ulcorner A_1 \urcorner \supset \ulcorner A_2 \urcorner \xrightarrow{\ell} \ulcorner A_1 \urcorner \supset \ulcorner A_2 \urcorner$ (Rule (claims))

3. ℓ claims $\ulcorner A_1 \urcorner \supset \ulcorner A_2 \urcorner \xrightarrow{\ell} \ell$ says $(\ulcorner A_1 \urcorner \supset \ulcorner A_2 \urcorner)$ (Rule (saysR))

4. ℓ claims $\ulcorner A_1 \urcorner \supset \ulcorner A_2 \urcorner, \ell$ says $(\ulcorner A_1 \urcorner \supset \ulcorner A_2 \urcorner) \xrightarrow{K} \ell$ says ℓ says $(\ulcorner A_1 \urcorner \supset \ulcorner A_2 \urcorner)$ (Rule (saysR))

5. ℓ says $(\ulcorner A_1 \urcorner \supset \ulcorner A_2 \urcorner) \xrightarrow{K} \ell$ says ℓ says $(\ulcorner A_1 \urcorner \supset \ulcorner A_2 \urcorner)$ (Rule (saysL))

Case. $A = K$ says B.

1. ℓ claims K says $\ulcorner B \urcorner, K$ says $\ulcorner B \urcorner \xrightarrow{\ell} K$ says $\ulcorner B \urcorner$ (Theorem B.3)

2. ℓ claims K says $\ulcorner B \urcorner \xrightarrow{\ell} K$ says $\ulcorner B \urcorner$ (Rule (claims))

3. ℓ claims K says $\ulcorner B \urcorner \xrightarrow{\ell} \ell$ says K says $\ulcorner B \urcorner$ (Rule (saysR))

4. ℓ claims K says $\ulcorner B \urcorner, \ell$ says K says $\ulcorner B \urcorner \xrightarrow{K} \ell$ says ℓ says K says $\ulcorner B \urcorner$ (Rule (saysR))

5. ℓ says K says $\ulcorner B \urcorner \xrightarrow{K} \ell$ says ℓ says K says $\ulcorner B \urcorner$ (Rule (saysL))

□

Now we prove soundness of the translation. If Γ is a set of DTL_0 formulas, we use the notation ℓ claims Γ to denote the DTL_0 hypothesis obtained by prefixing each formula in Γ with ℓ claims.

Lemma F.5 (Soundness of Translation). *The following hold for any ICL formula A, any ICL principal K, and any set Γ of ICL formulas.*

1. *If $\Gamma \vdash A$ in ICL's sequent calculus, then ℓ claims $\ulcorner \Gamma \urcorner \xrightarrow{\ell} \ulcorner A \urcorner$ in DTL_0.*

2. *If $\Gamma \vdash K$ affirms A in ICL's sequent calculus, then ℓ claims $\ulcorner \Gamma \urcorner \xrightarrow{K} \ulcorner A \urcorner$ in DTL_0.*

Proof. We prove both statements by simultaneous induction on the given proofs in ICL's sequent calculus. We analyze cases of the last rule.

Proof of (1).

Case. $\dfrac{(P \text{ atomic})}{\Gamma, P \vdash P}\text{init}$

1. ℓ claims $\ulcorner \Gamma \urcorner, \ell$ claims ℓ says P, ℓ says P, ℓ claims $P, P \xrightarrow{\ell} P$ (Rule (init))

2. ℓ claims $\ulcorner \Gamma \urcorner, \ell$ claims ℓ says P, ℓ says P, ℓ claims $P \xrightarrow{\ell} P$ (Rule (claims))

3. ℓ claims $\ulcorner \Gamma \urcorner, \ell$ claims ℓ says P, ℓ says $P \xrightarrow{\ell} P$ (Rule (saysL))

4. ℓ claims $\ulcorner \Gamma \urcorner, \ell$ claims ℓ says $P \xrightarrow{\ell} P$ (Rule (claims))

5. ℓ claims $\ulcorner \Gamma \urcorner, \ell$ claims ℓ says $P \xrightarrow{\ell} \ell$ says P (Rule (saysR))

Case. $\dfrac{\Gamma \vdash K \mathsf{~affirms~} A}{\Gamma \vdash K \mathsf{~says~} A}\text{saysR}$

1. $\ell \mathsf{~claims~} \ulcorner\Gamma\urcorner \xrightarrow{K} \ulcorner A\urcorner$ (i.h. on premise)

2. $\ell \mathsf{~claims~} \ulcorner\Gamma\urcorner \xrightarrow{\ell} K \mathsf{~says~} \ulcorner A\urcorner$ (Rule (saysR))

3. $\ell \mathsf{~claims~} \ulcorner\Gamma\urcorner \xrightarrow{\ell} \ell \mathsf{~says~} K \mathsf{~says~} \ulcorner A\urcorner$ (Rule (saysR))

Case. $\dfrac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \wedge B}\wedge\text{R}$

1. $\ell \mathsf{~claims~} \ulcorner\Gamma\urcorner \xrightarrow{\ell} \ulcorner A\urcorner$ (i.h. on 1st premise)

2. $\ell \mathsf{~claims~} \ulcorner\Gamma\urcorner \xrightarrow{\ell} \ulcorner B\urcorner$ (i.h. on 2nd premise)

3. $\ell \mathsf{~claims~} \ulcorner\Gamma\urcorner \xrightarrow{\ell} \ulcorner A\urcorner \wedge \ulcorner B\urcorner$ (Rule ($\wedge$R))

Case. $\dfrac{\Gamma, A \wedge B, A, B \vdash C}{\Gamma, A \wedge B \vdash C}\wedge\text{L}$

1. $\ell \mathsf{~claims~} \ulcorner\Gamma\urcorner, \ell \mathsf{~claims~} (\ulcorner A\urcorner \wedge \ulcorner B\urcorner), \ell \mathsf{~claims~} \ulcorner A\urcorner, \ell \mathsf{~claims~} \ulcorner B\urcorner \xrightarrow{\ell} \ulcorner C\urcorner$
   (i.h. on premise)

2. $\ell \mathsf{~claims~} \ulcorner\Gamma\urcorner, \ell \mathsf{~claims~} (\ulcorner A\urcorner \wedge \ulcorner B\urcorner), \ell \mathsf{~says~} \ulcorner A\urcorner, \ell \mathsf{~says~} \ulcorner B\urcorner \xrightarrow{\ell} \ulcorner C\urcorner$
   (Weakening and Rule (saysL))

3. $\ulcorner A\urcorner \xrightarrow{\ell} \ell \mathsf{~says~} \ulcorner A\urcorner$ (Lemma F.4)

4. $\ell \mathsf{~claims~} \ulcorner\Gamma\urcorner, \ell \mathsf{~claims~} (\ulcorner A\urcorner \wedge \ulcorner B\urcorner), \ulcorner A\urcorner, \ell \mathsf{~says~} \ulcorner B\urcorner \xrightarrow{\ell} \ulcorner C\urcorner$
   (Theorem B.2 on 3 and 2)

5. $\ulcorner B\urcorner \xrightarrow{\ell} \ell \mathsf{~says~} \ulcorner B\urcorner$ (Lemma F.4)

6. $\ell \mathsf{~claims~} \ulcorner\Gamma\urcorner, \ell \mathsf{~claims~} (\ulcorner A\urcorner \wedge \ulcorner B\urcorner), \ulcorner A\urcorner, \ulcorner B\urcorner \xrightarrow{\ell} \ulcorner C\urcorner$ (Theorem B.2 on 5 and 4)

7. $\ell \mathsf{~claims~} \ulcorner\Gamma\urcorner, \ell \mathsf{~claims~} (\ulcorner A\urcorner \wedge \ulcorner B\urcorner), \ulcorner A\urcorner \wedge \ulcorner B\urcorner, \ulcorner A\urcorner, \ulcorner B\urcorner \xrightarrow{\ell} \ulcorner C\urcorner$ (Weakening)

8. $\ell \mathsf{~claims~} \ulcorner\Gamma\urcorner, \ell \mathsf{~claims~} (\ulcorner A\urcorner \wedge \ulcorner B\urcorner), \ulcorner A\urcorner \wedge \ulcorner B\urcorner \xrightarrow{\ell} \ulcorner C\urcorner$ (Rule ($\wedge$L))

9. $\ell \mathsf{~claims~} \ulcorner\Gamma\urcorner, \ell \mathsf{~claims~} (\ulcorner A\urcorner \wedge \ulcorner B\urcorner) \xrightarrow{\ell} \ulcorner C\urcorner$ (Rule (claims))

Case. $\dfrac{\Gamma \vdash A}{\Gamma \vdash A \vee B}\vee\text{R}_1$

1. $\ell \mathsf{~claims~} \ulcorner\Gamma\urcorner \xrightarrow{\ell} \ulcorner A\urcorner$ (i.h. on premise)

2. ℓ claims $\ulcorner\Gamma\urcorner \xrightarrow{\ell} \ulcorner A\urcorner \vee \ulcorner B\urcorner$ (Rule ($\vee$ R$_1$))

Case. $\dfrac{\Gamma \vdash B}{\Gamma \vdash A \vee B}\vee \mathrm{R}_2$

1. ℓ claims $\ulcorner\Gamma\urcorner \xrightarrow{\ell} \ulcorner B\urcorner$ (i.h. on premise)
2. ℓ claims $\ulcorner\Gamma\urcorner \xrightarrow{\ell} \ulcorner A\urcorner \vee \ulcorner B\urcorner$ (Rule ($\vee$ R$_2$))

Case. $\dfrac{\Gamma, A \vee B, A \vdash C \qquad \Gamma, A \vee B, B \vdash C}{\Gamma, A \vee B \vdash C}\vee\mathrm{L}$

1. ℓ claims $\ulcorner\Gamma\urcorner, \ell$ claims $(\ulcorner A\urcorner \vee \ulcorner B\urcorner), \ell$ claims $\ulcorner A\urcorner \xrightarrow{\ell} \ulcorner C\urcorner$ (i.h. on 1st premise)
2. ℓ claims $\ulcorner\Gamma\urcorner, \ell$ claims $(\ulcorner A\urcorner \vee \ulcorner B\urcorner), \ell$ says $\ulcorner A\urcorner \xrightarrow{\ell} \ulcorner C\urcorner$

   (Weakening and Rule (saysL))
3. $\ulcorner A\urcorner \xrightarrow{\ell} \ell$ says $\ulcorner A\urcorner$ (Lemma F.4)
4. ℓ claims $\ulcorner\Gamma\urcorner, \ell$ claims $(\ulcorner A\urcorner \vee \ulcorner B\urcorner), \ulcorner A\urcorner \xrightarrow{\ell} \ulcorner C\urcorner$ (Theorem B.2 on 3 and 2)
5. ℓ claims $\ulcorner\Gamma\urcorner, \ell$ claims $(\ulcorner A\urcorner \vee \ulcorner B\urcorner), \ulcorner A\urcorner \vee \ulcorner B\urcorner, \ulcorner A\urcorner \xrightarrow{\ell} \ulcorner C\urcorner$ (Weakening)
6. ℓ claims $\ulcorner\Gamma\urcorner, \ell$ claims $(\ulcorner A\urcorner \vee \ulcorner B\urcorner), \ulcorner A\urcorner \vee \ulcorner B\urcorner, \ulcorner B\urcorner \xrightarrow{\ell} \ulcorner C\urcorner$ (Similar to 5)
7. ℓ claims $\ulcorner\Gamma\urcorner, \ell$ claims $(\ulcorner A\urcorner \vee \ulcorner B\urcorner), \ulcorner A\urcorner \vee \ulcorner B\urcorner \xrightarrow{\ell} \ulcorner C\urcorner$ (Rule ($\vee$L) on 5 and 6)
8. ℓ claims $\ulcorner\Gamma\urcorner, \ell$ claims $(\ulcorner A\urcorner \vee \ulcorner B\urcorner) \xrightarrow{\ell} \ulcorner C\urcorner$ (Rule (claims))

Case. $\dfrac{}{\Gamma \vdash \top}\top\mathrm{R}$

1. ℓ claims $\ulcorner\Gamma\urcorner \xrightarrow{\ell} \top$ (Rule ($\top$R))

Case. $\dfrac{}{\Gamma, \bot \vdash C}\bot\mathrm{L}$

1. ℓ claims $\ulcorner\Gamma\urcorner, \ell$ claims $\bot, \bot \xrightarrow{\ell} \ulcorner C\urcorner$ (Rule ($\bot$L))
2. ℓ claims $\ulcorner\Gamma\urcorner, \ell$ claims $\bot \xrightarrow{\ell} \ulcorner C\urcorner$ (Rule (claims))

Case. $\dfrac{\Gamma, A \vdash B}{\Gamma \vdash A \supset B}\supset\mathrm{R}$

1. ℓ claims $\ulcorner\Gamma\urcorner, \ell$ claims $\ulcorner A\urcorner \xrightarrow{\ell} \ulcorner B\urcorner$ (i.h. on premise)
2. ℓ claims $\ulcorner\Gamma\urcorner, \ell$ says $\ulcorner A\urcorner \xrightarrow{\ell} \ulcorner B\urcorner$ (Weakening and Rule (saysL))

3. $\ulcorner A \urcorner \xrightarrow{\ell} \ell$ says $\ulcorner A \urcorner$ (Lemma F.4)

4. ℓ claims $\ulcorner \Gamma \urcorner, \ulcorner A \urcorner \xrightarrow{\ell} \ulcorner B \urcorner$ (Theorem B.2 on 3 and 2)

5. ℓ claims $\ulcorner \Gamma \urcorner \xrightarrow{\ell} \ulcorner A \urcorner \supset \ulcorner B \urcorner$ (Rule ($\supset$R))

6. ℓ claims $\ulcorner \Gamma \urcorner \xrightarrow{\ell} \ell$ says $(\ulcorner A \urcorner \supset \ulcorner B \urcorner)$ (Rule (saysR))

Case. $\dfrac{\Gamma, A \supset B \vdash A \qquad \Gamma, A \supset B, B \vdash C}{\Gamma, A \supset B \vdash C}\supset\text{L}$

1. ℓ claims $\ulcorner \Gamma \urcorner, \ell$ claims ℓ says $(\ulcorner A \urcorner \supset \ulcorner B \urcorner) \xrightarrow{\ell} \ulcorner A \urcorner$ (i.h. on 1st premise)

2. ℓ claims $\ulcorner \Gamma \urcorner, \ell$ claims ℓ says $(\ulcorner A \urcorner \supset \ulcorner B \urcorner), \ell$ claims $\ulcorner B \urcorner \xrightarrow{\ell} \ulcorner C \urcorner$
(i.h. on 2nd premise)

3. ℓ claims $\ulcorner \Gamma \urcorner, \ell$ claims ℓ says $(\ulcorner A \urcorner \supset \ulcorner B \urcorner), \ell$ says $\ulcorner B \urcorner \xrightarrow{\ell} \ulcorner C \urcorner$
(Weakening and Rule (saysL))

4. $\ulcorner B \urcorner \xrightarrow{\ell} \ell$ says $\ulcorner B \urcorner$ (Lemma F.4)

5. ℓ claims $\ulcorner \Gamma \urcorner, \ell$ claims ℓ says $(\ulcorner A \urcorner \supset \ulcorner B \urcorner), \ulcorner B \urcorner \xrightarrow{\ell} \ulcorner C \urcorner$ (Theorem B.2 on 4 and 3)

6. ℓ claims $\ulcorner \Gamma \urcorner, \ell$ claims ℓ says $(\ulcorner A \urcorner \supset \ulcorner B \urcorner), \ulcorner A \urcorner \supset \ulcorner B \urcorner \xrightarrow{\ell} \ulcorner C \urcorner$
(Rule ($\supset$L) on 1 and 5)

7. ℓ claims $\ulcorner \Gamma \urcorner, \ell$ claims ℓ says $(\ulcorner A \urcorner \supset \ulcorner B \urcorner), \ell$ says $(\ulcorner A \urcorner \supset \ulcorner B \urcorner), \ell$ claims $(\ulcorner A \urcorner \supset \ulcorner B \urcorner), \ulcorner A \urcorner \supset \ulcorner B \urcorner \xrightarrow{\ell} \ulcorner C \urcorner$ (Weakening)

8. ℓ claims $\ulcorner \Gamma \urcorner, \ell$ claims ℓ says $(\ulcorner A \urcorner \supset \ulcorner B \urcorner), \ell$ says $(\ulcorner A \urcorner \supset \ulcorner B \urcorner), \ell$ claims $(\ulcorner A \urcorner \supset \ulcorner B \urcorner) \xrightarrow{\ell} \ulcorner C \urcorner$ (Rule (claims))

9. ℓ claims $\ulcorner \Gamma \urcorner, \ell$ claims ℓ says $(\ulcorner A \urcorner \supset \ulcorner B \urcorner), \ell$ says $(\ulcorner A \urcorner \supset \ulcorner B \urcorner) \xrightarrow{\ell} \ulcorner C \urcorner$
(Rule (saysL))

10. ℓ claims $\ulcorner \Gamma \urcorner, \ell$ claims ℓ says $(\ulcorner A \urcorner \supset \ulcorner B \urcorner) \xrightarrow{\ell} \ulcorner C \urcorner$ (Rule (claims))

Proof of (2).

Case. $\dfrac{\Gamma \vdash A}{\Gamma \vdash K \text{ affirms } A}\text{affs}$

1. ℓ claims $\ulcorner \Gamma \urcorner \xrightarrow{\ell} \ulcorner A \urcorner$ (i.h. (1) on premise)

2. ℓ claims $\ulcorner\Gamma\urcorner \xrightarrow{K} \ulcorner A\urcorner$ (Theorem B.1 on 1)

Case. $\dfrac{\Gamma, K \textsf{ says } A, A \vdash K \textsf{ affirms } C}{\Gamma, K \textsf{ says } A \vdash K \textsf{ affirms } C}\text{saysL}$

1. ℓ claims $\ulcorner\Gamma\urcorner, \ell$ claims ℓ says K says $\ulcorner A\urcorner, \ell$ claims $\ulcorner A\urcorner \xrightarrow{K} \ulcorner C\urcorner$ (i.h. on premise)
2. ℓ claims $\ulcorner\Gamma\urcorner, \ell$ claims ℓ says K says $\ulcorner A\urcorner, \ell$ says $\ulcorner A\urcorner \xrightarrow{K} \ulcorner C\urcorner$

   (Weakning and Rule (saysL))
3. $\ulcorner A\urcorner \xrightarrow{K} \ell$ says $\ulcorner A\urcorner$ (Lemma F.4)
4. ℓ claims $\ulcorner\Gamma\urcorner, \ell$ claims ℓ says K says $\ulcorner A\urcorner, \ulcorner A\urcorner \xrightarrow{K} \ulcorner C\urcorner$ (Theorem B.2 on 3, 2)
5. ℓ claims $\ulcorner\Gamma\urcorner, \ell$ claims ℓ says K says $\ulcorner A\urcorner, \ell$ says K says $\ulcorner A\urcorner, \ell$ claims K says $\ulcorner A\urcorner, K$ says $\ulcorner A\urcorner, K$ claims $\ulcorner A\urcorner, \ulcorner A\urcorner \xrightarrow{K} \ulcorner C\urcorner$ (Weakening)
6. ℓ claims $\ulcorner\Gamma\urcorner, \ell$ claims ℓ says K says $\ulcorner A\urcorner, \ell$ says K says $\ulcorner A\urcorner, \ell$ claims K says $\ulcorner A\urcorner, K$ says $\ulcorner A\urcorner, K$ claims $\ulcorner A\urcorner \xrightarrow{K} \ulcorner C\urcorner$ (Rule (claims))
7. ℓ claims $\ulcorner\Gamma\urcorner, \ell$ claims ℓ says K says $\ulcorner A\urcorner, \ell$ says K says $\ulcorner A\urcorner, \ell$ claims K says $\ulcorner A\urcorner, K$ says $\ulcorner A\urcorner \xrightarrow{K} \ulcorner C\urcorner$ (Rule (saysL))
8. ℓ claims $\ulcorner\Gamma\urcorner, \ell$ claims ℓ says K says $\ulcorner A\urcorner, \ell$ says K says $\ulcorner A\urcorner, \ell$ claims K says $\ulcorner A\urcorner \xrightarrow{K} \ulcorner C\urcorner$ (Rule (claims))
9. ℓ claims $\ulcorner\Gamma\urcorner, \ell$ claims ℓ says K says $\ulcorner A\urcorner, \ell$ says K says $\ulcorner A\urcorner \xrightarrow{K} \ulcorner C\urcorner$ (Rule (saysL))
10. ℓ claims $\ulcorner\Gamma\urcorner, \ell$ claims ℓ says K says $\ulcorner A\urcorner \xrightarrow{K} \ulcorner C\urcorner$ (Rule (claims))

Case. $\dfrac{\Gamma, A \wedge B, A, B \vdash K \textsf{ affirms } C}{\Gamma, A \wedge B \vdash K \textsf{ affirms } C}\wedge\text{Laff}$

1. ℓ claims $\ulcorner\Gamma\urcorner, \ell$ claims $(\ulcorner A\urcorner \wedge \ulcorner B\urcorner), \ell$ claims $\ulcorner A\urcorner, \ell$ claims $\ulcorner B\urcorner \xrightarrow{K} \ulcorner C\urcorner$

   (i.h. on premise)
2. ℓ claims $\ulcorner\Gamma\urcorner, \ell$ claims $(\ulcorner A\urcorner \wedge \ulcorner B\urcorner), \ell$ says $\ulcorner A\urcorner, \ell$ says $\ulcorner B\urcorner \xrightarrow{K} \ulcorner C\urcorner$

   (Weakening and Rule (saysL))
3. $\ulcorner A\urcorner \xrightarrow{K} \ell$ says $\ulcorner A\urcorner$ (Lemma F.4)
4. ℓ claims $\ulcorner\Gamma\urcorner, \ell$ claims $(\ulcorner A\urcorner \wedge \ulcorner B\urcorner), \ulcorner A\urcorner, \ell$ says $\ulcorner B\urcorner \xrightarrow{K} \ulcorner C\urcorner$

   (Theorem B.2 on 3 and 2)
5. $\ulcorner B\urcorner \xrightarrow{K} \ell$ says $\ulcorner B\urcorner$ (Lemma F.4)

6. ℓ claims $\ulcorner\Gamma\urcorner, \ell$ claims $(\ulcorner A\urcorner \wedge \ulcorner B\urcorner), \ulcorner A\urcorner, \ulcorner B\urcorner \xrightarrow{K} \ulcorner C\urcorner$ (Theorem B.2 on 5 and 4)

7. ℓ claims $\ulcorner\Gamma\urcorner, \ell$ claims $(\ulcorner A\urcorner \wedge \ulcorner B\urcorner), \ulcorner A\urcorner \wedge \ulcorner B\urcorner, \ulcorner A\urcorner, \ulcorner B\urcorner \xrightarrow{K} \ulcorner C\urcorner$ (Weakening)

8. ℓ claims $\ulcorner\Gamma\urcorner, \ell$ claims $(\ulcorner A\urcorner \wedge \ulcorner B\urcorner), \ulcorner A\urcorner \wedge \ulcorner B\urcorner \xrightarrow{K} \ulcorner C\urcorner$ (Rule ($\wedge$L))

9. ℓ claims $\ulcorner\Gamma\urcorner, \ell$ claims $(\ulcorner A\urcorner \wedge \ulcorner B\urcorner) \xrightarrow{K} \ulcorner C\urcorner$ (Rule (claims))

Case. $\dfrac{\Gamma, A \vee B, A \vdash K \text{ affirms } C \qquad \Gamma, A \vee B, B \vdash K \text{ affirms } C}{\Gamma, A \vee B \vdash K \text{ affirms } C}\vee\text{Laff}$

1. ℓ claims $\ulcorner\Gamma\urcorner, \ell$ claims $(\ulcorner A\urcorner \vee \ulcorner B\urcorner), \ell$ claims $\ulcorner A\urcorner \xrightarrow{K} \ulcorner C\urcorner$ (i.h. on 1st premise)

2. ℓ claims $\ulcorner\Gamma\urcorner, \ell$ claims $(\ulcorner A\urcorner \vee \ulcorner B\urcorner), \ell$ says $\ulcorner A\urcorner \xrightarrow{K} \ulcorner C\urcorner$

   (Weakening and Rule (saysL))

3. $\ulcorner A\urcorner \xrightarrow{K} \ell$ says $\ulcorner A\urcorner$ (Lemma F.4)

4. ℓ claims $\ulcorner\Gamma\urcorner, \ell$ claims $(\ulcorner A\urcorner \vee \ulcorner B\urcorner), \ulcorner A\urcorner \xrightarrow{K} \ulcorner C\urcorner$ (Theorem B.2 on 3 and 2)

5. ℓ claims $\ulcorner\Gamma\urcorner, \ell$ claims $(\ulcorner A\urcorner \vee \ulcorner B\urcorner), \ulcorner A\urcorner \vee \ulcorner B\urcorner, \ulcorner A\urcorner \xrightarrow{K} \ulcorner C\urcorner$ (Weakening)

6. ℓ claims $\ulcorner\Gamma\urcorner, \ell$ claims $(\ulcorner A\urcorner \vee \ulcorner B\urcorner), \ulcorner A\urcorner \vee \ulcorner B\urcorner, \ulcorner B\urcorner \xrightarrow{K} \ulcorner C\urcorner$ (Similar to 5)

7. ℓ claims $\ulcorner\Gamma\urcorner, \ell$ claims $(\ulcorner A\urcorner \vee \ulcorner B\urcorner), \ulcorner A\urcorner \vee \ulcorner B\urcorner \xrightarrow{K} \ulcorner C\urcorner$ (Rule ($\vee$L) on 5 and 6)

8. ℓ claims $\ulcorner\Gamma\urcorner, \ell$ claims $(\ulcorner A\urcorner \vee \ulcorner B\urcorner) \xrightarrow{K} \ulcorner C\urcorner$ (Rule (claims))

Case. $\dfrac{}{\Gamma, \bot \vdash K \text{ affirms } C}\bot\text{Laff}$

1. ℓ claims $\ulcorner\Gamma\urcorner, \ell$ claims $\bot, \bot \xrightarrow{K} \ulcorner C\urcorner$ (Rule ($\bot$L))

2. ℓ claims $\ulcorner\Gamma\urcorner, \ell$ claims $\bot \xrightarrow{K} \ulcorner C\urcorner$ (Rule (claims))

Case. $\dfrac{\Gamma, A \supset B \vdash A \qquad \Gamma, A \supset B, B \vdash K \text{ affirms } C}{\Gamma, A \supset B \vdash K \text{ affirms } C}\supset\text{Laff}$

1. ℓ claims $\ulcorner\Gamma\urcorner, \ell$ claims ℓ says $(\ulcorner A\urcorner \supset \ulcorner B\urcorner) \xrightarrow{\ell} \ulcorner A\urcorner$ (i.h. on 1st premise)

2. ℓ claims $\ulcorner\Gamma\urcorner, \ell$ claims ℓ says $(\ulcorner A\urcorner \supset \ulcorner B\urcorner) \xrightarrow{K} \ulcorner A\urcorner$ (Theorem B.1 on 1)

3. ℓ claims $\ulcorner\Gamma\urcorner, \ell$ claims ℓ says $(\ulcorner A\urcorner \supset \ulcorner B\urcorner), \ell$ claims $\ulcorner B\urcorner \xrightarrow{K} \ulcorner C\urcorner$

   (i.h. on 2nd premise)

4. ℓ claims $\ulcorner\Gamma\urcorner, \ell$ claims ℓ says $(\ulcorner A\urcorner \supset \ulcorner B\urcorner), \ell$ says $\ulcorner B\urcorner \xrightarrow{K} \ulcorner C\urcorner$

   (Weakening and Rule (saysL))

5. $\ulcorner B \urcorner \xrightarrow{K} \ell\ \mathsf{says}\ \ulcorner B \urcorner$ (Lemma F.4)

6. $\ell\ \mathsf{claims}\ \ulcorner \Gamma \urcorner, \ell\ \mathsf{claims}\ \ell\ \mathsf{says}\ (\ulcorner A \urcorner \supset \ulcorner B \urcorner), \ulcorner B \urcorner \xrightarrow{K} \ulcorner C \urcorner$ (Theorem B.2 on 5 and 4)

7. $\ell\ \mathsf{claims}\ \ulcorner \Gamma \urcorner, \ell\ \mathsf{claims}\ \ell\ \mathsf{says}\ (\ulcorner A \urcorner \supset \ulcorner B \urcorner), \ulcorner A \urcorner \supset \ulcorner B \urcorner \xrightarrow{K} \ulcorner C \urcorner$

   (Rule ($\supset$L) on 2 and 6)

8. $\ell\ \mathsf{claims}\ \ulcorner \Gamma \urcorner, \ell\ \mathsf{claims}\ \ell\ \mathsf{says}\ (\ulcorner A \urcorner \supset \ulcorner B \urcorner), \ell\ \mathsf{says}\ (\ulcorner A \urcorner \supset \ulcorner B \urcorner), \ell\ \mathsf{claims}\ (\ulcorner A \urcorner \supset \ulcorner B \urcorner), \ulcorner A \urcorner \supset \ulcorner B \urcorner \xrightarrow{K} \ulcorner C \urcorner$ (Weakening)

9. $\ell\ \mathsf{claims}\ \ulcorner \Gamma \urcorner, \ell\ \mathsf{claims}\ \ell\ \mathsf{says}\ (\ulcorner A \urcorner \supset \ulcorner B \urcorner), \ell\ \mathsf{says}\ (\ulcorner A \urcorner \supset \ulcorner B \urcorner), \ell\ \mathsf{claims}\ (\ulcorner A \urcorner \supset \ulcorner B \urcorner) \xrightarrow{K} \ulcorner C \urcorner$ (Rule (claims))

10. $\ell\ \mathsf{claims}\ \ulcorner \Gamma \urcorner, \ell\ \mathsf{claims}\ \ell\ \mathsf{says}\ (\ulcorner A \urcorner \supset \ulcorner B \urcorner), \ell\ \mathsf{says}\ (\ulcorner A \urcorner \supset \ulcorner B \urcorner) \xrightarrow{K} \ulcorner C \urcorner$

    (Rule (saysL))

11. $\ell\ \mathsf{claims}\ \ulcorner \Gamma \urcorner, \ell\ \mathsf{claims}\ \ell\ \mathsf{says}\ (\ulcorner A \urcorner \supset \ulcorner B \urcorner) \xrightarrow{K} \ulcorner C \urcorner$ (Rule (claims))

□

F.2 Proof of Completeness

We prove completeness of the translation by a method of simulation between proofs. First we characterize syntactically the sequents that can arise in a proof of a translated formula. We call such sequents regular sequents. Next, we define an inverse translation ($\llcorner \cdot \lrcorner$) from regular sequents to sequents of ICL, and prove by induction that any proof in DTL_0 that ends in a regular sequent can also be simulated (under the inverse translation) in ICL. From this completeness follows immediately. As a convention, we use the letter k to denote a principal from ICL, and K to denote a principal from DTL_0. The latter may either be a principal from ICL or ℓ.

Definition F.6 (Regular Sequents)**.** A DTL_0 hypothesis Γ is called regular if it contains assumptions of the following forms only: $\ulcorner A \urcorner$, $k\ \mathsf{claims}\ \ulcorner A \urcorner$, $k\ \mathsf{says}\ \ulcorner A \urcorner$, $\ulcorner A \urcorner \supset \ulcorner B \urcorner$, P, $\ell\ \mathsf{claims}\ k\ \mathsf{says}\ \ulcorner A \urcorner$, $\ell\ \mathsf{claims}\ \ulcorner A \urcorner \supset \ulcorner B \urcorner$, or $\ell\ \mathsf{claims}\ P$.

A DTL_0 sequent is called regular if it falls into one of the two cateories:

1. (α-regular) $\Gamma \xrightarrow{k} \ulcorner C \urcorner$, where Γ is a regular hypothesis.

2. (β-regular) $\Gamma \xrightarrow{\ell} C$, where Γ is a regular hypothesis, and C has one of the forms $\ulcorner A \urcorner$, $k\ \mathsf{says}\ \ulcorner A \urcorner$, $\ulcorner A \urcorner \supset \ulcorner B \urcorner$, or P.

Definition F.7 (Inverse Translation)**.** The inverse translation for regular hypothesis Γ (written $\llcorner \Gamma \lrcorner$) is defined pointwise, where the inverse translation for assumptions is

defined as follows:

$$\begin{array}{rcl}
\llcorner\ulcorner A\urcorner\lrcorner & = & A\\
\llcorner k \textsf{ claims } \ulcorner A\urcorner\lrcorner & = & k \textsf{ says } A\\
\llcorner k \textsf{ says } \ulcorner A\urcorner\lrcorner & = & k \textsf{ says } A\\
\llcorner\ulcorner A\urcorner \supset \ulcorner B\urcorner\lrcorner & = & A \supset B\\
\llcorner P\lrcorner & = & P\\
\llcorner \ell \textsf{ claims } k \textsf{ says } \ulcorner A\urcorner\lrcorner & = & k \textsf{ says } A\\
\llcorner \ell \textsf{ claims } \ulcorner A\urcorner \supset \ulcorner B\urcorner\lrcorner & = & A \supset B\\
\llcorner \ell \textsf{ claims } P\lrcorner & = & P
\end{array}$$

The inverse translation for the α-regular sequent $\Gamma \xrightarrow{k} \ulcorner C\urcorner$ is defined as $\llcorner\Gamma\lrcorner \vdash K \textsf{ affirms } C$. The inverse translation for the β-regular sequent $\Gamma \xrightarrow{\ell} C$ is defined as $\llcorner\Gamma\lrcorner \vdash \llcorner C\lrcorner$, where the inverse translation of C is defined as follows:

$$\begin{array}{rcl}
\llcorner\ulcorner A\urcorner\lrcorner & = & A\\
\llcorner k \textsf{ says } \ulcorner A\urcorner\lrcorner & = & k \textsf{ says } A\\
\llcorner\ulcorner A\urcorner \supset \ulcorner B\urcorner\lrcorner & = & A \supset B\\
\llcorner P\lrcorner & = & P
\end{array}$$

Lemma F.8 (Completeness of Translation)**.** *The following hold.*

1. *If $\Gamma \xrightarrow{k} \ulcorner C\urcorner$ is α-regular and provable in DTL_0, then $\llcorner\Gamma\lrcorner \vdash K \textsf{ affirms } C$ in ICL's sequent calculus.*

2. *If $\Gamma \xrightarrow{\ell} C$ is β-regular and provable in DTL_0, then $\llcorner\Gamma\lrcorner \vdash \llcorner C\lrcorner$ in ICL's sequent calculus.*

Proof. We prove both statements by a simultaneous induction on the depths of the given derivations, and case analyze the last rule in the derivations.

Proof of (1).

Case. $$\dfrac{P \text{ atomic}}{\Gamma, P \xrightarrow{k} P}\text{init}$$

1. $\llcorner\Gamma\lrcorner, P \vdash P$ (Rule (init))
2. $\llcorner\Gamma\lrcorner, P \vdash k \textsf{ affirms } P$ (Rule (affs))

Case. $$\dfrac{\Gamma, k \textsf{ claims } \ulcorner A\urcorner, \ulcorner A\urcorner \xrightarrow{k} \ulcorner C\urcorner \qquad k \succeq k}{\Gamma, k \textsf{ claims } \ulcorner A\urcorner \xrightarrow{k} \ulcorner C\urcorner}\text{claims}$$

1. $\llcorner\Gamma\lrcorner, k \textsf{ says } A, A \vdash k \textsf{ affirms } C$ (i.h. on premise)
2. $\llcorner\Gamma\lrcorner, k \textsf{ says } A \vdash k \textsf{ affirms } C$ (Rule (saysL)

Case. $$\dfrac{\Gamma, \ell \textsf{ claims } A, A \xrightarrow{k} \ulcorner C\urcorner \qquad \ell \succeq k}{\Gamma, \ell \textsf{ claims } A \xrightarrow{k} \ulcorner C\urcorner}\text{claims}$$

By regularity, A must have one of the forms $k \textsf{ says } \ulcorner B\urcorner$, $\ulcorner B_1\urcorner \supset \ulcorner B_2\urcorner$, or P. It is easy to check that in each case, $\llcorner \ell \textsf{ claims } A\lrcorner = \llcorner A\lrcorner$. Thus we have:

1. $\llcorner\Gamma\lrcorner, \llcorner\ell \text{ claims } A\lrcorner, \llcorner A\lrcorner \vdash k \text{ affirms } C$ (i.h. on premise)

2. $\llcorner\Gamma\lrcorner, \llcorner\ell \text{ claims } A\lrcorner \vdash k \text{ affirms } C$ (Strengthening; $\llcorner\ell \text{ claims } A\lrcorner = \llcorner A\lrcorner$)

Case. $$\frac{\Gamma|_\ell \xrightarrow{\ell} C}{\Gamma \xrightarrow{k} \ell \text{ says } C}\text{saysR}$$

By regularity, $\ell \text{ says } C$ must have form $\ulcorner C'\urcorner$, and hence C must have one of the forms $k' \text{ says } \ulcorner A\urcorner$, $\ulcorner A\urcorner \supset \ulcorner B\urcorner$, or P. In each case, observe that the premise is β-regular, and that $\llcorner C\lrcorner = \llcorner\ell \text{ says } C\lrcorner$. Thus we have

1. $\llcorner\Gamma|_\ell\lrcorner \vdash \llcorner C\lrcorner$ (i.h. (2) on premise)

2. $\llcorner\Gamma\lrcorner \vdash \llcorner C\lrcorner$ (Weakening)

3. $\llcorner\Gamma\lrcorner \vdash k \text{ affirms } \llcorner C\lrcorner$ (Rule (affs))

4. $\llcorner\Gamma\lrcorner \vdash k \text{ affirms } \llcorner\ell \text{ says } C\lrcorner$ ($\llcorner C\lrcorner = \llcorner\ell \text{ says } C\lrcorner$)

Case. $$\frac{\Gamma, k' \text{ says } \ulcorner A\urcorner, k' \text{ claims } \ulcorner A\urcorner \xrightarrow{k} \ulcorner C\urcorner}{\Gamma, k' \text{ says } \ulcorner A\urcorner \xrightarrow{k} \ulcorner C\urcorner}\text{saysL}$$

1. $\llcorner\Gamma\lrcorner, k' \text{ says } A, k' \text{ says } A \vdash k \text{ affirms } C$ (i.h. on premise)

2. $\llcorner\Gamma\lrcorner, k' \text{ says } A \vdash k \text{ affirms } C$ (Strengthening)

Case. $$\frac{\Gamma, \ell \text{ says } A, \ell \text{ claims } A \xrightarrow{k} \ulcorner C\urcorner}{\Gamma, \ell \text{ says } A \xrightarrow{k} \ulcorner C\urcorner}\text{saysL}$$

By regularity, A must have one of the forms $k' \text{ says } \ulcorner A'\urcorner$, $\ulcorner B\urcorner \supset \ulcorner C\urcorner$, or P. In each case, observe that the premise is also α-regular, and that $\llcorner\ell \text{ says } A\lrcorner = \llcorner\ell \text{ claims } A\lrcorner$.

1. $\llcorner\Gamma\lrcorner, \llcorner\ell \text{ says } A\lrcorner, \llcorner\ell \text{ claims } A\lrcorner \vdash k \text{ affirms } C$ (i.h. on premise)

2. $\llcorner\Gamma\lrcorner, \llcorner\ell \text{ says } A\lrcorner \vdash k \text{ affirms } C$ (Strengthening; $\llcorner\ell \text{ says } A\lrcorner = \llcorner\ell \text{ claims } A\lrcorner$)

Case. $$\frac{\Gamma \xrightarrow{k} \ulcorner A\urcorner \qquad \Gamma \xrightarrow{k} \ulcorner B\urcorner}{\Gamma \xrightarrow{k} \ulcorner A\urcorner \wedge \ulcorner B\urcorner}\wedge\text{R}$$

1. $\llcorner\Gamma\lrcorner \vdash k \text{ affirms } A$ (i.h. on 1st premise)

2. $\llcorner\Gamma\lrcorner \vdash k \text{ affirms } B$ (i.h. on 2nd premise)

3. $A, B \vdash A \wedge B$ (Provable in ICL)

4. $A, B \vdash k \text{ affirms } (A \wedge B)$ (Rule (affs))

5. $\llcorner\Gamma\lrcorner, A \vdash k \text{ affirms } (A \wedge B)$ (Lemma F.2.2 on 2 and 4)

6. $\llcorner\Gamma\lrcorner \vdash k \text{ affirms } (A \wedge B)$ (Lemma F.2.2 on 1 and 5)

Case. $$\dfrac{\Gamma, \ulcorner A \urcorner \wedge \ulcorner B \urcorner, \ulcorner A \urcorner, \ulcorner B \urcorner \xrightarrow{k} \ulcorner C \urcorner}{\Gamma, \ulcorner A \urcorner \wedge \ulcorner B \urcorner \xrightarrow{k} \ulcorner C \urcorner}\wedge \mathrm{L}$$

1. $\llcorner \Gamma \lrcorner, A \wedge B, A, B \vdash k$ affirms C (i.h. on premise)
2. $\llcorner \Gamma \lrcorner, A \wedge B \vdash k$ affirms C (Rule ($\wedge$Laff))

Case. $$\dfrac{\Gamma \xrightarrow{k} \ulcorner A \urcorner}{\Gamma \xrightarrow{k} \ulcorner A \urcorner \vee \ulcorner B \urcorner}\vee \mathrm{R}_1$$

1. $\llcorner \Gamma \lrcorner \vdash k$ affirms A (i.h. on premise)
2. $A \vdash A \vee B$ (Provable in ICL)
3. $A \vdash k$ affirms $(A \vee B)$ (Rule (affs))
4. $\llcorner \Gamma \lrcorner \vdash k$ affirms $(A \vee B)$ (Lemma F.2.2 on 1 and 3)

Case. $$\dfrac{\Gamma \xrightarrow{k} \ulcorner B \urcorner}{\Gamma \xrightarrow{k} \ulcorner A \urcorner \vee \ulcorner B \urcorner}\vee \mathrm{R}_2$$

1. $\llcorner \Gamma \lrcorner \vdash k$ affirms B (i.h. on premise)
2. $B \vdash A \vee B$ (Provable in ICL)
3. $B \vdash k$ affirms $(A \vee B)$ (Rule (affs))
4. $\llcorner \Gamma \lrcorner \vdash k$ affirms $(A \vee B)$ (Lemma F.2.2 on 1 and 3)

Case. $$\dfrac{\Gamma, \ulcorner A \urcorner \vee \ulcorner B \urcorner, \ulcorner A \urcorner \xrightarrow{k} \ulcorner C \urcorner \qquad \Gamma, \ulcorner A \urcorner \vee \ulcorner B \urcorner, \ulcorner B \urcorner \xrightarrow{k} \ulcorner C \urcorner}{\Gamma, \ulcorner A \urcorner \vee \ulcorner B \urcorner \xrightarrow{k} \ulcorner C \urcorner}\vee \mathrm{L}$$

1. $\llcorner \Gamma \lrcorner, A \vee B, A \vdash k$ affirms C (i.h. on 1st premise)
2. $\llcorner \Gamma \lrcorner, A \vee B, B \vdash k$ affirms C (i.h. on 2nd premise)
3. $\llcorner \Gamma \lrcorner, A \vee B \vdash k$ affirms C (Rule ($\vee$Laff))

Case. $$\dfrac{}{\Gamma \xrightarrow{k} \top}\top \mathrm{R}$$

1. $\llcorner \Gamma \lrcorner \vdash \top$ (Rule ($\top$R))
2. $\llcorner \Gamma \lrcorner \vdash k$ affirms $\top$ (Rule (affs))

Case. $$\dfrac{}{\Gamma, \bot \xrightarrow{k} \ulcorner C \urcorner}\bot \mathrm{L}$$

1. $\llcorner \Gamma \lrcorner, \bot \vdash k$ affirms C (Rule ($\bot$Laff))

Case. Rule ($\supset$R) does not arise.

Case. $$\dfrac{\Gamma, \ulcorner A \urcorner \supset \ulcorner B \urcorner \xrightarrow{k} \ulcorner A \urcorner \qquad \Gamma, \ulcorner A \urcorner \supset \ulcorner B \urcorner, \ulcorner B \urcorner \xrightarrow{k} \ulcorner C \urcorner}{\Gamma, \ulcorner A \urcorner \supset \ulcorner B \urcorner \xrightarrow{k} \ulcorner C \urcorner}\supset\text{L}$$

1. $\llcorner \Gamma \lrcorner, A \supset B \vdash k \textsf{ affirms } A$ (i.h. on 1st premise)
2. $\llcorner \Gamma \lrcorner, A \supset B, B \vdash k \textsf{ affirms } C$ (i.h. on 2nd premise)
3. $\llcorner \Gamma \lrcorner, A \supset B, B, A \vdash k \textsf{ affirms } C$ (Weakening)
4. $\llcorner \Gamma \lrcorner, A \supset B, A \vdash A$ (Lemma F.3)
5. $\llcorner \Gamma \lrcorner, A \supset B, A \vdash k \textsf{ affirms } C$ (Rule ($\supset$Laff) on 4 and 3)
6. $\llcorner \Gamma \lrcorner, A \supset B \vdash k \textsf{ affirms } C$ (Lemma F.2 on 1 and 5)

Proof of (2).

Case. $$\dfrac{P \text{ atomic}}{\Gamma, P \xrightarrow{\ell} P}\text{init}$$

1. $\llcorner \Gamma \lrcorner, P \vdash P$ (Rule (init))

Case. $$\dfrac{\Gamma, \ell \textsf{ claims } A, A \xrightarrow{\ell} C \qquad \ell \succeq \ell}{\Gamma, \ell \textsf{ claims } A \xrightarrow{\ell} C}\text{claims}$$

By regularity, A must have one of the forms $k \textsf{ says } \ulcorner B \urcorner$, $\ulcorner B_1 \urcorner \supset \ulcorner B_2 \urcorner$, or P. It is easy to check that in each case, $\llcorner \ell \textsf{ claims } A \lrcorner = \llcorner A \lrcorner$. Thus we have:

1. $\llcorner \Gamma \lrcorner, \llcorner \ell \textsf{ claims } A \lrcorner, \llcorner A \lrcorner \vdash \llcorner C \lrcorner$ (i.h. on premise)
2. $\llcorner \Gamma \lrcorner, \llcorner \ell \textsf{ claims } A \lrcorner \vdash \llcorner C \lrcorner$ (Strengthening; $\llcorner \ell \textsf{ claims } A \lrcorner = \llcorner A \lrcorner$)

Case. $$\dfrac{\Gamma|_{\ell} \xrightarrow{\ell} C}{\Gamma \xrightarrow{\ell} \ell \textsf{ says } C}\textsf{saysR}$$

By regularity, $\ell \textsf{ says } C$ must have form $\ulcorner C' \urcorner$, and hence C must have one of the forms $k' \textsf{ says } \ulcorner A \urcorner$, $\ulcorner A \urcorner \supset \ulcorner B \urcorner$, or P. In each case, observe that the premise is also β-regular, and that $\llcorner C \lrcorner = \llcorner \ell \textsf{ says } C \lrcorner$. Thus we have

1. $\llcorner \Gamma|_{\ell} \lrcorner \vdash \llcorner C \lrcorner$ (i.h. on premise)
2. $\llcorner \Gamma \lrcorner \vdash \llcorner C \lrcorner$ (Weakening)
3. $\llcorner \Gamma \lrcorner \vdash \llcorner \ell \textsf{ says } C \lrcorner$ ($\llcorner C \lrcorner = \llcorner \ell \textsf{ says } C \lrcorner$)

Case. $\dfrac{\Gamma|_k \xrightarrow{k} \ulcorner C \urcorner}{\Gamma \xrightarrow{\ell} k \textsf{ says } \ulcorner C \urcorner}\textsf{saysR}$

Observe that the premise is α-regular.

1. $\llcorner \Gamma|_k \lrcorner \vdash k \textsf{ affirms } C$ (i.h. (1) on premise)
2. $\llcorner \Gamma|_k \lrcorner \vdash k \textsf{ says } C$ (Rule (saysR))
3. $\llcorner \Gamma \lrcorner \vdash k \textsf{ says } C$ (Weakening)

Case. $\dfrac{\Gamma, k' \textsf{ says } \ulcorner A \urcorner, k' \textsf{ claims } \ulcorner A \urcorner \xrightarrow{\ell} C}{\Gamma, k' \textsf{ says } \ulcorner A \urcorner \xrightarrow{\ell} C}\textsf{saysL}$

1. $\llcorner \Gamma \lrcorner, k' \textsf{ says } A, k' \textsf{ says } A \vdash \llcorner C \lrcorner$ (i.h. on premise)
2. $\llcorner \Gamma \lrcorner, k' \textsf{ says } A \vdash \llcorner C \lrcorner$ (Strengthening)

Case. $\dfrac{\Gamma, \ell \textsf{ says } A, \ell \textsf{ claims } A \xrightarrow{\ell} C}{\Gamma, \ell \textsf{ says } A \xrightarrow{\ell} C}\textsf{saysL}$

By regularity, A must have one of the forms $k' \textsf{ says } \ulcorner A' \urcorner$, $\ulcorner B \urcorner \supset \ulcorner B' \urcorner$, or P. In each case, observe that the premise is also β-regular, and that $\llcorner \ell \textsf{ says } A \lrcorner = \llcorner \ell \textsf{ claims } A \lrcorner$.

1. $\llcorner \Gamma \lrcorner, \llcorner \ell \textsf{ says } A \lrcorner, \llcorner \ell \textsf{ claims } A \lrcorner \vdash \llcorner C \lrcorner$ (i.h. on premise)
2. $\llcorner \Gamma \lrcorner, \llcorner \ell \textsf{ says } A \lrcorner \vdash \llcorner C \lrcorner$ (Strengthening; $\llcorner \ell \textsf{ says } A \lrcorner = \llcorner \ell \textsf{ claims } A \lrcorner$)

Case. $\dfrac{\Gamma \xrightarrow{\ell} \ulcorner A \urcorner \qquad \Gamma \xrightarrow{\ell} \ulcorner B \urcorner}{\Gamma \xrightarrow{\ell} \ulcorner A \urcorner \wedge \ulcorner B \urcorner}\wedge\mathrm{R}$

1. $\llcorner \Gamma \lrcorner \vdash A$ (i.h. on 1st premise)
2. $\llcorner \Gamma \lrcorner \vdash B$ (i.h. on 2nd premise)
3. $\llcorner \Gamma \lrcorner \vdash (A \wedge B)$ (Rule ($\wedge$R))

Case. $\dfrac{\Gamma, \ulcorner A \urcorner \wedge \ulcorner B \urcorner, \ulcorner A \urcorner, \ulcorner B \urcorner \xrightarrow{\ell} C}{\Gamma, \ulcorner A \urcorner \wedge \ulcorner B \urcorner \xrightarrow{\ell} C}\wedge\mathrm{L}$

1. $\llcorner \Gamma \lrcorner, A \wedge B, A, B \vdash \llcorner C \lrcorner$ (i.h. on premise)
2. $\llcorner \Gamma \lrcorner, A \wedge B \vdash \llcorner C \lrcorner$ (Rule ($\wedge$L))

Case. $\dfrac{\Gamma \xrightarrow{\ell} \ulcorner A \urcorner}{\Gamma \xrightarrow{\ell} \ulcorner A \urcorner \vee \ulcorner B \urcorner}\vee\mathrm{R}_1$

1. $\llcorner \Gamma \lrcorner \vdash A$ (i.h. on premise)

2. $\llcorner\Gamma\lrcorner \vdash A \vee B$ (Rule ($\vee$ R$_1$))

Case. $$\dfrac{\Gamma \xrightarrow{\ell} \ulcorner B \urcorner}{\Gamma \xrightarrow{\ell} \ulcorner A \urcorner \vee \ulcorner B \urcorner}\vee \mathrm{R}_2$$

1. $\llcorner\Gamma\lrcorner \vdash B$ (i.h. on premise)
2. $\llcorner\Gamma\lrcorner \vdash A \vee B$ (Rule ($\vee$ R$_2$))

Case. $$\dfrac{\Gamma, \ulcorner A \urcorner \vee \ulcorner B \urcorner, \ulcorner A \urcorner \xrightarrow{\ell} C \qquad \Gamma, \ulcorner A \urcorner \vee \ulcorner B \urcorner, \ulcorner B \urcorner \xrightarrow{\ell} C}{\Gamma, \ulcorner A \urcorner \vee \ulcorner B \urcorner \xrightarrow{\ell} C}\vee \mathrm{L}$$

1. $\llcorner\Gamma\lrcorner, A \vee B, A \vdash \llcorner C \lrcorner$ (i.h. on 1st premise)
2. $\llcorner\Gamma\lrcorner, A \vee B, B \vdash \llcorner C \lrcorner$ (i.h. on 2nd premise)
3. $\llcorner\Gamma\lrcorner, A \vee B \vdash \llcorner C \lrcorner$ (Rule ($\vee$L))

Case. $$\dfrac{}{\Gamma \xrightarrow{\ell} \top}\top\mathrm{R}$$

1. $\llcorner\Gamma\lrcorner \vdash \top$ (Rule ($\top$R))

Case. $$\dfrac{}{\Gamma, \bot \xrightarrow{\ell} C}\bot\mathrm{L}$$

1. $\llcorner\Gamma\lrcorner, \bot \vdash \llcorner C \lrcorner$ (Rule ($\bot$L))

Case. $$\dfrac{\Gamma, \ulcorner A \urcorner \xrightarrow{\ell} \ulcorner B \urcorner}{\Gamma \xrightarrow{\ell} \ulcorner A \urcorner \supset \ulcorner B \urcorner}\supset\mathrm{R}$$

1. $\llcorner\Gamma\lrcorner, A \vdash B$ (i.h. on premise)
2. $\llcorner\Gamma\lrcorner \vdash A \supset B$ (Rule ($\supset$R))

Case. $$\dfrac{\Gamma, \ulcorner A \urcorner \supset \ulcorner B \urcorner \xrightarrow{\ell} \ulcorner A \urcorner \qquad \Gamma, \ulcorner A \urcorner \supset \ulcorner B \urcorner, \ulcorner B \urcorner \xrightarrow{\ell} C}{\Gamma, \ulcorner A \urcorner \supset \ulcorner B \urcorner \xrightarrow{\ell} C}\supset\mathrm{L}$$

1. $\llcorner\Gamma\lrcorner, A \supset B \vdash A$ (i.h. on 1st premise)
2. $\llcorner\Gamma\lrcorner, A \supset B, B \vdash \llcorner C \lrcorner$ (i.h. on 2nd premise)
3. $\llcorner\Gamma\lrcorner, A \supset B \vdash \llcorner C \lrcorner$ (Rule ($\supset$L))

□

Theorem F.9 (Correctness; Theorem 5.3)**.** *$\vdash A$ in ICL if and only if $\cdot \xrightarrow{\ell} \ulcorner A \urcorner$ in DTL$_0$.*

Proof. Suppose $\vdash A$ in ICL. By Lemma F.1, $\cdot \vdash A$ in ICL's sequent calculus. Thus by Lemma F.5.1, $\cdot \xrightarrow{\ell} \ulcorner A \urcorner$ in DTL$_0$.

Conversely, suppose that $\cdot \xrightarrow{\ell} \ulcorner A \urcorner$ in DTL$_0$. By Lemma F.8.2, $\cdot \vdash A$ in ICL's sequent calculus. By Lemma F.1, $\vdash A$ in ICL's axiomatic system. □

G Proofs from Section 5.3

In this appendix we prove that the translation from IIK to DTL_0 is both sound and complete (Theorem 5.4). Part of our proof uses a generalized axiomatic proof system for IIK, which we develop first.

G.1 The Axiomatic System for IIK

In Section 5.3, we listed the axioms and rules of IIK that are specific to the modality K says A. Below we list *all* the rules and axioms of IIK.

$$\frac{\vdash A}{\vdash K \;\textsf{says}\; A}\text{nec} \qquad \frac{\vdash A \supset B \qquad \vdash A}{\vdash B}\text{mp} \qquad \frac{A \text{ is an axiom}}{\vdash A}\text{ax}$$

Axioms:

$$\begin{array}{ll}
(K \;\textsf{says}\; (A \supset B)) \supset ((K \;\textsf{says}\; A) \supset (K \;\textsf{says}\; B)) & \text{(K)} \\
A \supset (B \supset A) & \text{(imp1)} \\
(A \supset B) \supset ((A \supset (B \supset C)) \supset (A \supset C)) & \text{(imp2)} \\
A \supset (B \supset (A \wedge B)) & \text{(conj1)} \\
(A \wedge B) \supset A & \text{(conj2)} \\
(A \wedge B) \supset B & \text{(conj3)} \\
A \supset (A \vee B) & \text{(disj1)} \\
B \supset (A \vee B) & \text{(disj2)} \\
(A \supset C) \supset ((B \supset C) \supset ((A \vee B) \supset C)) & \text{(disj3)} \\
\top & \text{(true)} \\
\bot \supset A & \text{(false)}
\end{array}$$

Next, we generalize the axiomatic system, adding hypothetical reasoning, as we did for the axiomatic system of DTL_0 (Appendix C). We write $\Gamma \vdash_G A$ to mean that A follows from the formulas in Γ. The rules of deduction are:

$$\frac{}{\Gamma, A \vdash_G A}\text{use} \qquad \frac{\cdot \vdash_G A}{\Gamma \vdash_G K \;\textsf{says}\; A}\text{nec} \qquad \frac{\Gamma \vdash_G A \supset B \qquad \Gamma \vdash_G A}{\Gamma \vdash_G B}\text{mp}$$

$$\frac{A \text{ is an axiom}}{\Gamma \vdash_G A}\text{ax}$$

As for DTL_0, we prove some elementary properties for the generalized system of IIK, and show also that the generalized system and axiomatic system are equivalent.

Lemma G.1 (Basic properties)**.** *The following hold.*

1. *(Weakening)* $\Gamma \vdash_G A$ *implies* $\Gamma, \Gamma' \vdash_G A$
2. *(Substitution)* $\Gamma \vdash_G A$ *and* $\Gamma, A \vdash_G B$ *imply* $\Gamma \vdash_G B$

Proof. Exactly as for DTL_0 in Lemma C.1, since the proof does not rely on the specific axioms used. □

Theorem G.2 (Deduction)**.** *The following hold.*

1. $\Gamma \vdash_G A \supset B$ *implies* $\Gamma, A \vdash_G B$

2. $\Gamma, A \vdash_G B$ *implies* $\Gamma \vdash_G A \supset B$

Proof. Exactly as for DTL_0 in Theorem C.2. The proof does not rely on the axioms (4), (C), or (S), which are the only axiom present in DTL_0 that are not present in IIK. □

Theorem G.3 (G iff Axiomatic). $\vdash A$ *if and only if* $\cdot \vdash_G A$

Proof. In each direction by straightforward induction on the given derivation. □

G.2 Proof of Soundness

Lemma G.4 (Soundness of Translation). *If* $\vdash A$ *in IIK, then for each* K, $\cdot \xrightarrow{K} \ulcorner A \urcorner$ *in ICL's axiomatic system.*

Proof. We induct on the derivation of $\vdash A$, analyzing cases of the last rule in it.

Case. $\dfrac{\vdash A}{\vdash K' \text{ says } A}\text{nec}$

1. $\cdot \xrightarrow{K'} \ulcorner A \urcorner$ (i.h. on premise)
2. $\cdot \xrightarrow{d} K' \text{ says } \ulcorner A \urcorner$ (Rule (saysR))
3. $\cdot \xrightarrow{K} d \text{ says } K' \text{ says } \ulcorner A \urcorner$ (Rule (saysR))

Case. $\dfrac{\vdash A \supset B \qquad \vdash A}{\vdash B}\text{mp}$

1. $\cdot \xrightarrow{K} \ulcorner A \urcorner \supset \ulcorner B \urcorner$ (i.h. on 1st premise)
2. $\cdot \xrightarrow{K} \ulcorner A \urcorner$ (i.h. on 2nd premise)
3. $\ulcorner B \urcorner \xrightarrow{K} \ulcorner B \urcorner$ (Theorem B.3)
4. $\ulcorner A \urcorner \supset \ulcorner B \urcorner \xrightarrow{K} \ulcorner B \urcorner$ (Rule ($\supset$L) on 2 and 3)
5. $\cdot \xrightarrow{K} \ulcorner B \urcorner$ (Theorem B.2 on 1 and 4)

Case. $\dfrac{A \text{ is an axiom}}{\vdash A}\text{ax}$

We case analyze all possible axioms A.

Case. (Axiom K) $A = (K' \text{ says } (A' \supset B')) \supset ((K' \text{ says } A') \supset (K' \text{ says } B'))$

1. $\ulcorner A' \urcorner \supset \ulcorner B' \urcorner, \ulcorner A' \urcorner \xrightarrow{K'} \ulcorner B' \urcorner$ (Provable in DTL_0)
2. $K' \text{ claims } (\ulcorner A' \urcorner \supset \ulcorner B' \urcorner), K' \text{ claims } \ulcorner A' \urcorner \xrightarrow{K'} \ulcorner B' \urcorner$

(Weakening and rule (claims))

3. K' claims $(\ulcorner A' \urcorner \supset \ulcorner B' \urcorner), K'$ claims $\ulcorner A' \urcorner \xrightarrow{d} K'$ says $\ulcorner B' \urcorner$ (Rule (saysR))
4. K' says $(\ulcorner A' \urcorner \supset \ulcorner B' \urcorner), K'$ says $\ulcorner A' \urcorner \xrightarrow{d} K'$ says $\ulcorner B' \urcorner$
(Weakening and rule (saysL))
5. d claims K' says $(\ulcorner A' \urcorner \supset \ulcorner B' \urcorner), d$ claims K' says $\ulcorner A' \urcorner \xrightarrow{d} K'$ says $\ulcorner B' \urcorner$
(Weakening and rule (claims))
6. d claims K' says $(\ulcorner A' \urcorner \supset \ulcorner B' \urcorner), d$ claims K' says $\ulcorner A' \urcorner \xrightarrow{K} d$ says K' says $\ulcorner B' \urcorner$
(Rule (saysR))
7. d says K' says $(\ulcorner A' \urcorner \supset \ulcorner B' \urcorner), d$ says K' says $\ulcorner A' \urcorner \xrightarrow{K} d$ says K' says $\ulcorner B' \urcorner$
(Weakening and rule (saysL))
8. $\cdot \xrightarrow{K} (d$ says K' says $(\ulcorner A' \urcorner \supset \ulcorner B' \urcorner)) \supset ((d$ says K' says $\ulcorner A' \urcorner) \supset d$ says K' says $\ulcorner B' \urcorner)$ (Rule ($\supset$R))

The remaining cases are straightforward.

□

G.3 Proof of Completeness

Our proof of completeness needs a basic lemma about proofs in DTL_0. We also use this lemma later to prove other translations correct.

Lemma G.5 (Inversion in DTL_0)**.** *The following hold in the sequent calculus of DTL_0.*

1. *If Γ, K says $A \xrightarrow{K'} C$ then Γ, K claims $A \xrightarrow{K'} C$ by a shorter or equal derivation.*
2. *If $\Gamma \xrightarrow{K} A \wedge B$ then $\Gamma \xrightarrow{K} A$ and $\Gamma \xrightarrow{K} B$ by shorter or equal derivations.*

Proof. In each case by induction on the given derivations. □

Next, we carefully characterize sequents that may occur in the proof of a translated formula. We call these sequents *regular*. As a general convention, we use the letter k to denote principals in IIK, and K to denote principals in DTL_0. The latter may either be principals from IIK, or d. The principal ℓ, although present in DTL_0, never shows up in proofs of translated formulas. This is a consequence of the subformula property of DTL_0's sequent calculus.

Definition G.6 (Regular Hypothesis)**.** Given an IIK principal k, we call a hypothesis Γ k-regular if the following holds:

1. Γ contains assumptions of the forms $\ulcorner A \urcorner$, k claims $\ulcorner A \urcorner$, and d claims k' says $\ulcorner A \urcorner$ only, where A denotes an arbitrary formula in IIK. (Note that the principal k in "k-regular" is the same as the principal k in k claims $\ulcorner A \urcorner$.)

We call a DTL_0 hypothesis Γ d-regular if the following hold:

1. Γ contains assumptions of the forms d claims k says $\ulcorner A \urcorner$ and k claims $\ulcorner A \urcorner$ only, where A denotes an arbitrary formula in IIK.

2. k claims $\ulcorner A \urcorner \in \Gamma$ implies d claims k says $\ulcorner A \urcorner \in \Gamma$.

Definition G.7 (Regular Sequent)**.** We call a sequent regular if it has one of the following forms:

1. (k-regular) $\Gamma \xrightarrow{k} \ulcorner A \urcorner$, where Γ is k-regular.

2. (d-regular) $\Gamma \xrightarrow{d} k$ says $\ulcorner A \urcorner$, where Γ is d-regular.

Next we define an inverse translation $\llcorner \cdot \lrcorner_K$ from regular hypothesis to hypothesis in IIK's generalized axiomatic system.

Definition G.8 (Inverse Translation)**.** The inverse translation $\llcorner \Gamma \lrcorner_k$ for a k-regular hypothesis Γ is defined pointwise, where the inverse translation for assumptions is:

$$\begin{array}{rcl} \llcorner \ulcorner A \urcorner \lrcorner_k & = & A \\ \llcorner k \text{ claims } \ulcorner A \urcorner \lrcorner_k & = & A \\ \llcorner d \text{ claims } k' \text{ says } \ulcorner A \urcorner \lrcorner_k & = & k' \text{ says } A \end{array}$$

The inverse translation $\llcorner \Gamma \lrcorner_d$ for a d-regular hypothesis Γ is defined pointwise, where the inverse translation for assumptions is:

$$\begin{array}{rcl} \llcorner d \text{ claims } k \text{ says } \ulcorner A \urcorner \lrcorner_d & = & k \text{ says } A \\ \llcorner k \text{ claims } \ulcorner A \urcorner \lrcorner_d & = & \cdot \ \text{(Empty)} \end{array}$$

Lemma G.9 (Completeness of Translation)**.** *The following hold:*

1. *If $\Gamma \xrightarrow{k} \ulcorner A \urcorner$ is k-regular and provable in DTL_0, then $\llcorner \Gamma \lrcorner_k \vdash_G A$ is provable in IIK's generalized axiomatic system.*

2. *If $\Gamma \xrightarrow{d} k$ says $\ulcorner A \urcorner$ is d-regular and provable in DTL_0, then $\llcorner \Gamma \lrcorner_d \vdash_G k$ says A is provable in IIK's generalized axiomatic system.*

Proof. We prove both statements simultaneously by induction on the *depths* of the given derivations. In each case, we analyze the last rule in the derivation.

Proof of (1).

Case. $\dfrac{P \text{ atomic}}{\Gamma, P \xrightarrow{k} P}\text{init}$

1. $\llcorner \Gamma \lrcorner_k, P \vdash_G P$ (Rule (use))

Case. $\dfrac{\Gamma, k \text{ claims } \ulcorner A \urcorner, \ulcorner A \urcorner \xrightarrow{k} \ulcorner B \urcorner \qquad k \succeq k}{\Gamma, k \text{ claims } \ulcorner A \urcorner \xrightarrow{k} \ulcorner B \urcorner}\text{claims}$

1. $\llcorner \Gamma \lrcorner_k, A, A \vdash_G B$ (i.h. on premise)

2. $\llcorner\Gamma\lrcorner_k, A \vdash_G A$ (Rule (use))

3. $\llcorner\Gamma\lrcorner_k, A \vdash_G B$ (Lemma G.1.2 on 2 and 1)

Case. $$\dfrac{\Gamma|_d \xrightarrow{d} k' \,\mathsf{says}\, \ulcorner A \urcorner}{\Gamma \xrightarrow{k} d \,\mathsf{says}\, k' \,\mathsf{says}\, \ulcorner A \urcorner}\,\mathsf{says}\mathrm{R}$$

By k-regularity of Γ, $\Gamma|_d$ must have the form d claims k_1 says $\ulcorner B_1 \urcorner, \ldots, d$ claims k_n says $\ulcorner B_n \urcorner$. Clearly, $\Gamma|_d$ is d-regular. Hence the premise is d-regular.

1. $\llcorner\Gamma|_d\lrcorner_d \vdash_G k' \,\mathsf{says}\, A$ (i.h. (2) on premise)

2. $\llcorner\Gamma|_d\lrcorner_d \subseteq \llcorner\Gamma\lrcorner_k$ (Defn.)

3. $\llcorner\Gamma\lrcorner_k \vdash_G k' \,\mathsf{says}\, A$ (Lemma G.1.1 on 1 using 2)

Case. $$\dfrac{\Gamma, d \,\mathsf{says}\, k' \,\mathsf{says}\, \ulcorner A \urcorner, d \,\mathsf{claims}\, k' \,\mathsf{says}\, \ulcorner A \urcorner \xrightarrow{k} \ulcorner C \urcorner}{\Gamma, d \,\mathsf{says}\, k' \,\mathsf{says}\, \ulcorner A \urcorner \xrightarrow{k} \ulcorner C \urcorner}\,\mathsf{saysL}$$

1. $\llcorner\Gamma\lrcorner_k, k' \,\mathsf{says}\, A, k' \,\mathsf{says}\, A \vdash_G C$ (i.h. on premise)

2. $\llcorner\Gamma\lrcorner_k, k' \,\mathsf{says}\, A \vdash_G k' \,\mathsf{says}\, A$ (Rule (use))

3. $\llcorner\Gamma\lrcorner_k, k' \,\mathsf{says}\, A \vdash_G C$ (Lemma G.1.2 on 2 and 1)

Case. $$\dfrac{\Gamma \xrightarrow{k} \ulcorner A \urcorner \qquad \Gamma \xrightarrow{k} \ulcorner B \urcorner}{\Gamma \xrightarrow{k} \ulcorner A \urcorner \wedge \ulcorner B \urcorner}\,\wedge\mathrm{R}$$

1. $\llcorner\Gamma\lrcorner_k \vdash_G A$ (i.h. on 1st premise)

2. $\llcorner\Gamma\lrcorner_k \vdash_G B$ (i.h. on 2nd premise)

3. $\llcorner\Gamma\lrcorner_k \vdash_G A \supset (B \supset (A \wedge B))$ (Rule (ax) and (conj1))

4. $\llcorner\Gamma\lrcorner_k \vdash_G B \supset (A \wedge B)$ (Rule (mp) on 3 and 1)

5. $\llcorner\Gamma\lrcorner_k \vdash_G A \wedge B$ (Rule (mp) on 4 and 2)

Case. $$\dfrac{\Gamma, \ulcorner A \urcorner \wedge \ulcorner B \urcorner, \ulcorner A \urcorner, \ulcorner B \urcorner \xrightarrow{k} \ulcorner C \urcorner}{\Gamma, \ulcorner A \urcorner \wedge \ulcorner B \urcorner \xrightarrow{k} \ulcorner C \urcorner}\,\wedge\mathrm{L}$$

1. $\llcorner\Gamma\lrcorner_k, A \wedge B, A, B \vdash_G C$ (i.h. on premise)

2. $\llcorner\Gamma\lrcorner_k, A \wedge B \vdash_G A \wedge B$ (Rule (use))

3. $\llcorner\Gamma\lrcorner_k, A \wedge B \vdash_G (A \wedge B) \supset A$ (Rule (ax) and (conj2))

4. $\llcorner\Gamma\lrcorner_k, A \wedge B \vdash_G A$ (Rule (mp) on 3 and 2)

5. $\llcorner\Gamma\lrcorner_k, A \wedge B \vdash_G (A \wedge B) \supset B$ (Rule (ax) and (conj3))

6. $\llcorner\Gamma\lrcorner_k, A \wedge B \vdash_G B$ (Rule (mp) on 5 and 2)

7. $\llcorner\Gamma\lrcorner_k, A \wedge B, B \vdash_G C$ (Lemma G.1.2 on 4 and 1)

8. $\llcorner\Gamma\lrcorner_k, A \wedge B \vdash_G C$ (Lemma G.1.2 on 6 and 7)

Case. $\dfrac{\Gamma \xrightarrow{k} \ulcorner A \urcorner}{\Gamma \xrightarrow{k} \ulcorner A \urcorner \vee \ulcorner B \urcorner} \vee \mathrm{R}_1$

1. $\llcorner\Gamma\lrcorner_k \vdash_G A$ (i.h. on premise)
2. $\llcorner\Gamma\lrcorner_k \vdash_G A \supset (A \vee B)$ (Rule (ax) and (disj1))
3. $\llcorner\Gamma\lrcorner_k \vdash_G A \vee B$ (Rule (mp) on 2 and 1)

Case. $\dfrac{\Gamma \xrightarrow{k} \ulcorner B \urcorner}{\Gamma \xrightarrow{k} \ulcorner A \urcorner \vee \ulcorner B \urcorner} \vee \mathrm{R}_2$

1. $\llcorner\Gamma\lrcorner_k \vdash_G B$ (i.h. on premise)
2. $\llcorner\Gamma\lrcorner_k \vdash_G B \supset (A \vee B)$ (Rule (ax) and (disj2))
3. $\llcorner\Gamma\lrcorner_k \vdash_G A \vee B$ (Rule (mp) on 2 and 1)

Case. $\dfrac{\Gamma, \ulcorner A \urcorner \vee \ulcorner B \urcorner, \ulcorner A \urcorner \xrightarrow{k} \ulcorner C \urcorner \qquad \Gamma, \ulcorner A \urcorner \vee \ulcorner B \urcorner, \ulcorner B \urcorner \xrightarrow{k} \ulcorner C \urcorner}{\Gamma, \ulcorner A \urcorner \vee \ulcorner B \urcorner \xrightarrow{k} \ulcorner C \urcorner} \vee \mathrm{L}$

1. $\llcorner\Gamma\lrcorner_k, A \vee B, A \vdash_G C$ (i.h. on 1st premise)
2. $\llcorner\Gamma\lrcorner_k, A \vee B, B \vdash_G C$ (i.h. on 2nd premise)
3. $\llcorner\Gamma\lrcorner_k, A \vee B \vdash_G A \supset C$ (Theorem G.2 on 1)
4. $\llcorner\Gamma\lrcorner_k, A \vee B \vdash_G B \supset C$ (Theorem G.2 on 2)
5. $\llcorner\Gamma\lrcorner_k, A \vee B \vdash_G (A \supset C) \supset ((B \supset C) \supset ((A \vee B) \supset C))$ (Rule (ax) and (disj3))
6. $\llcorner\Gamma\lrcorner_k, A \vee B \vdash_G (B \supset C) \supset ((A \vee B) \supset C)$ (Rule (mp) on 5 and 3)
7. $\llcorner\Gamma\lrcorner_k, A \vee B \vdash_G (A \vee B) \supset C$ (Rule (mp) on 6 and 4)
8. $\llcorner\Gamma\lrcorner_k, A \vee B, A \vee B \vdash_G C$ (Theorem G.2 on 7)
9. $\llcorner\Gamma\lrcorner_k, A \vee B \vdash_G A \vee B$ (Rule (use))
10. $\llcorner\Gamma\lrcorner_k, A \vee B \vdash_G C$ (Lemma G.1.2 on 9 and 8)

Case. $\dfrac{}{\Gamma \xrightarrow{k} \top} \top\mathrm{R}$

1. $\llcorner\Gamma\lrcorner_k \vdash_G \top$ (Rule (ax) and (true))

Case. $\dfrac{}{\Gamma, \bot \xrightarrow{k} \ulcorner C \urcorner}\bot\mathrm{L}$

1. $\llcorner \Gamma \lrcorner_k \vdash_G \bot \supset C$ (Rule (ax) and (false))
2. $\llcorner \Gamma \lrcorner_k, \bot \vdash_G C$ (Theorem G.2)

Case. $\dfrac{\Gamma, \ulcorner A \urcorner \xrightarrow{k} \ulcorner B \urcorner}{\Gamma \xrightarrow{k} \ulcorner A \urcorner \supset \ulcorner B \urcorner}{\supset}\mathrm{R}$

1. $\llcorner \Gamma \lrcorner_k, A \vdash_G B$ (i.h. on premise)
2. $\llcorner \Gamma \lrcorner_k \vdash_G A \supset B$ (Theorem G.2)

Case. $\dfrac{\Gamma, \ulcorner A \urcorner \supset \ulcorner B \urcorner \xrightarrow{k} \ulcorner A \urcorner \qquad \Gamma, \ulcorner A \urcorner \supset \ulcorner B \urcorner, \ulcorner B \urcorner \xrightarrow{k} \ulcorner C \urcorner}{\Gamma, \ulcorner A \urcorner \supset \ulcorner B \urcorner \xrightarrow{k} \ulcorner C \urcorner}{\supset}\mathrm{L}$

1. $\llcorner \Gamma \lrcorner_k, A \supset B \vdash_G A$ (i.h. on 1st premise)
2. $\llcorner \Gamma \lrcorner_k, A \supset B, B \vdash_G C$ (i.h. on 2nd premise)
3. $\llcorner \Gamma \lrcorner_k, A \supset B \vdash_G A \supset B$ (Rule (use))
4. $\llcorner \Gamma \lrcorner_k, A \supset B \vdash_G B$ (Rule (mp) on 3 and 1)
5. $\llcorner \Gamma \lrcorner_k, A \supset B \vdash_G C$ (Lemma G.1.2 on 4 and 2)

Proof of (2).

Case. $\dfrac{\Gamma, d \text{ claims } k' \text{ says } \ulcorner B \urcorner, k' \text{ says } \ulcorner B \urcorner \xrightarrow{d} k \text{ says } \ulcorner A \urcorner \qquad d \succeq d}{\Gamma, d \text{ claims } k' \text{ says } \ulcorner B \urcorner \xrightarrow{d} k \text{ says } \ulcorner A \urcorner}\text{claims}$

1. $\Gamma, d \text{ claims } k' \text{ says } \ulcorner B \urcorner, k' \text{ claims } \ulcorner B \urcorner \xrightarrow{d} k \text{ says } \ulcorner A \urcorner$ (Lemma G.5.1 on premise)
2. $\llcorner \Gamma \lrcorner_d, k' \text{ says } B \vdash_G k \text{ says } A$ (i.h. on 1; the sequent in 1 is d-regular)

Case. $\dfrac{\Gamma|_k \xrightarrow{k} \ulcorner A \urcorner}{\Gamma \xrightarrow{d} k \text{ says } \ulcorner A \urcorner}\text{saysR}$

By d-regularity, $\Gamma|_k$ must have the form $k \text{ claims } \ulcorner B_1 \urcorner, \ldots, k \text{ claims } \ulcorner B_n \urcorner$, where $d \text{ claims } k \text{ says } \ulcorner B_i \urcorner \in \Gamma$. Clearly, the premise is k-regular.

1. $B_1, \ldots, B_n \vdash_G A$ (i.h. (1) on premise)
2. $\cdot \vdash_G (B_1 \wedge \ldots \wedge B_n) \supset A$ (Theorem G.2)
3. $\cdot \vdash_G k \text{ says } ((B_1 \wedge \ldots \wedge B_n) \supset A)$ (Rule (nec))
4. $\cdot \vdash_G (k \text{ says } B_1 \wedge \ldots \wedge k \text{ says } B_n) \supset k \text{ says } A$ (Rule (ax), (K), and rule (mp))

5. k says $B_1, \ldots, k$ says $B_n \vdash_G k$ says A (Theorem G.2)

6. $\{k$ says $B_1, \ldots, k$ says $B_n\} \subseteq \llcorner \Gamma \lrcorner_d$ (Defn.; d claims k says $\ulcorner B_i \urcorner \in \Gamma$)

7. $\llcorner \Gamma \lrcorner_d \vdash_G k$ says A (Lemma G.1.1 on 5 using 6)

□

Theorem G.10 (Correctness; Theorem 5.4)**.** *$\vdash A$ in IIK if and only if $\cdot \xrightarrow{\ell} \ulcorner A \urcorner$ in DTL_0.*

Proof. Suppose $\vdash A$ in IIK. Then by Lemma G.4, $\cdot \xrightarrow{\ell} \ulcorner A \urcorner$.

Conversely, suppose that $\cdot \xrightarrow{\ell} \ulcorner A \urcorner$. Pick any principal k in IIK. By Theorem B.1, $\cdot \xrightarrow{k} \ulcorner A \urcorner$. Using Lemma G.9.1, we get $\cdot \vdash_G A$ in IIK's generalized axiomatic system. Finally, using Theorem G.3, we have $\vdash A$ in IIK. □

H Proofs from Section 5.4

In this appendix we prove that the translation from SL to DTL_0 is correct (Theorem 5.5). First we prove a lemma about proofs in SL that is needed to establish the theorem.

Lemma H.1 (Basic properties of SL)**.** *The following hold for the inference system of SL.*

1. *(Weakening) If $\Delta \vdash_\Gamma G$ then $\Delta, A \vdash_\Gamma G$.*
2. *(Substitution) If $\Delta \vdash_\Gamma P$ and $\Delta, P \vdash_\Gamma G$ then $\Delta \vdash_\Gamma G$.*

Proof. (1) follows by an induction on the derivation of $\Delta \vdash_\Gamma G$. (2) follows by an induction on the derivation of $\Delta, P \vdash_\Gamma G$. We show below the cases in this proof.

Case. $$\frac{(P' \leftarrow G_1, \ldots, G_n) \in \Delta, P \qquad (\Delta, P \vdash_\Gamma G_i)_{i \in \{1,\ldots,n\}}}{\Delta, P \vdash_\Gamma P'}\text{bc (Non-principal case)}$$

1. $(P' \leftarrow G_1, \ldots, G_n) \in \Delta$ (From 1st premise)
2. $(\Delta \vdash_\Gamma G_i)_{i \in \{1,\ldots,n\}}$ (i.h. on 2nd premise)
3. $\Delta \vdash_\Gamma P'$ (Rule (bc) on 1 and 2)

Case. $$\frac{P \in \Delta, P \qquad \text{(No other premise)}}{\Delta, P \vdash_\Gamma P}\text{bc (Principal Case)}$$

1. $\Delta \vdash_\Gamma P$ (Given assumption)

Case. $$\frac{(K' : \Delta') \in \Gamma \qquad \Delta' \vdash_\Gamma P'}{\Delta, P \vdash_\Gamma K' \text{ says } P'}\text{says}$$

1. $\Delta \vdash_\Gamma K'$ says P' (Rule (says) on the two premises)

□

H.1 Proof of Soundness

Now we prove soundness of the translation.

Lemma H.2 (Soundness of Translation)**.** *Suppose* $(K : \Delta) \in \Gamma$. *Then* $\Delta \vdash_\Gamma G$ *in SL implies that* $\ulcorner\Gamma\urcorner, \ulcorner\Delta\urcorner \xrightarrow{K} \ulcorner G\urcorner$ *in* DTL_0*.*

Proof. We induct on the derivation of $\Delta \vdash_\Gamma G$, and case analyze its last rule.

Case. $\dfrac{(P \leftarrow G_1, \ldots, G_n) \in \Delta \qquad (\Delta \vdash_\Gamma G_i)_{i\in\{1,\ldots,n\}}}{\Delta \vdash_\Gamma P}\text{bc}$

1. $((\ulcorner G_1\urcorner \wedge \ldots \wedge \ulcorner G_n\urcorner) \supset P) \in \ulcorner\Delta\urcorner$ (Assumption $(P \leftarrow G_1, \ldots, G_n) \in \Delta$)
2. $\ulcorner\Gamma\urcorner, \ulcorner\Delta\urcorner \xrightarrow{K} \ulcorner G_i\urcorner$ (i.h. on 2nd premise)
3. $\ulcorner\Gamma\urcorner, \ulcorner\Delta\urcorner \xrightarrow{K} \ulcorner G_1\urcorner \wedge \ldots \wedge \ulcorner G_n\urcorner$ (Rule ($\wedge$R))
4. $\ulcorner\Gamma\urcorner, \ulcorner\Delta\urcorner, P \xrightarrow{K} P$ (Rule (init))
5. $\ulcorner\Gamma\urcorner, \ulcorner\Delta\urcorner \xrightarrow{K} P$ (Rule ($\supset$L) on 3 and 4 using 1)

Case. $\dfrac{(K' : \Delta') \in \Gamma \qquad \Delta' \vdash_\Gamma P}{\Delta \vdash_\Gamma K' \;\textsf{says}\; P}\text{says}$

Let $\Gamma = (K_i : \Delta_i)_{i\in\{1,\ldots,m\}}$, where each $\Delta_i = B_{1,i}, \ldots, B_{n_i,i}$. Then $\ulcorner\Gamma\urcorner = (\ell \;\textsf{says}\; K_i \;\textsf{says}\; \ulcorner B_{1,i}\urcorner, \ldots, \ell \;\textsf{says}\; K_i \;\textsf{says}\; \ulcorner B_{n_i,i}\urcorner)_{i\in\{1,\ldots,m\}}$. Further assume that $K' = K_t$ (where $t \in \{1, \ldots, m\}$), so that $\Delta' = B_{1,t}, \ldots, B_{n_t,t}$.

1. $(\ell \;\textsf{says}\; K_i \;\textsf{says}\; \ulcorner B_{1,i}\urcorner, \ldots, \ell \;\textsf{says}\; K_i \;\textsf{says}\; \ulcorner B_{n_i,i}\urcorner)_{i\in\{1,\ldots,m\}}, \ulcorner B_{1,t}\urcorner, \ldots, \ulcorner B_{n_t,t}\urcorner \xrightarrow{K_t} P$ (i.h. on premise)
2. $(\ell \;\textsf{claims}\; K_i \;\textsf{says}\; \ulcorner B_{1,i}\urcorner, \ldots, \ell \;\textsf{claims}\; K_i \;\textsf{says}\; \ulcorner B_{n_i,i}\urcorner)_{i\in\{1,\ldots,m\}}, \ulcorner B_{1,t}\urcorner, \ldots, \ulcorner B_{n_t,t}\urcorner \xrightarrow{K_t} P$ (Lemma G.5.1)
3. $(\ell \;\textsf{claims}\; K_i \;\textsf{says}\; \ulcorner B_{1,i}\urcorner, \ldots, \ell \;\textsf{claims}\; K_i \;\textsf{says}\; \ulcorner B_{n_i,i}\urcorner)_{i\in\{1,\ldots,m\}} \xrightarrow{K_t} P$

   (Weakening, Rules (claims), (saysL) and (claims))
4. $(\ell \;\textsf{claims}\; K_i \;\textsf{says}\; \ulcorner B_{1,i}\urcorner, \ldots, \ell \;\textsf{claims}\; K_i \;\textsf{says}\; \ulcorner B_{n_i,i}\urcorner)_{i\in\{1,\ldots,m\}} \xrightarrow{\ell} K_t \;\textsf{says}\; P$

   (Rule (saysR))
5. $(\ell \;\textsf{claims}\; K_i \;\textsf{says}\; \ulcorner B_{1,i}\urcorner, \ldots, \ell \;\textsf{claims}\; K_i \;\textsf{says}\; \ulcorner B_{n_i,i}\urcorner)_{i\in\{1,\ldots,m\}}, \Delta \xrightarrow{K} \ell \;\textsf{says}\; K_t \;\textsf{says}\; P$ (Rule (saysR))
6. $(\ell \;\textsf{says}\; K_i \;\textsf{says}\; \ulcorner B_{1,i}\urcorner, \ldots, \ell \;\textsf{says}\; K_i \;\textsf{says}\; \ulcorner B_{n_i,i}\urcorner)_{i\in\{1,\ldots,m\}}, \Delta \xrightarrow{K} \ell \;\textsf{says}\; K_t \;\textsf{says}\; P$

   (Weakening and Rule (saysL))

□

H.2 Proof of Completeness

To prove completeness of the translation, we carefully characterize DTL_0 sequents that may occur in the proof of a translated Soutei query. We call these sequents regular sequents. As a general convention, we use the letter k to denote principals in SL, and the letter K to denote principals in DTL_0. The latter may either be principals from SL, or ℓ. In addition to Γ, we also use the letter Φ to denote DTL_0 hypothesis. (The categorical judgments allowed in the hypothesis denoted by the symbols Γ and Φ differ, as described below.)

Definition H.3 (Regular Hypothesis)**.** A DTL_0 hypothesis Γ is called 0-regular if the following hold:

1. All assumptions in Γ have the form ℓ claims k says $\ulcorner A \urcorner$ or k claims A, where A denotes an arbitrary SL clause.
2. k claims $\ulcorner A \urcorner \in \Gamma$ implies ℓ claims k says $\ulcorner A \urcorner \in \Gamma$

If k is a principal in SL, we call the DTL_0 hypothesis Γ, Φ k-regular if the following hold:

1. Γ is a 0-regular hypothesis.
2. All assumptions in Φ have the form P or $\ulcorner A \urcorner$, where P denotes an arbitrary atomic formula, and A denotes an arbitrary SL clause.
3. $\ulcorner A \urcorner \in \Phi$ and $A \neq P$ implies k claims $\ulcorner A \urcorner \in \Gamma$.

Definition H.4 (Regular Sequents)**.** We call a DTL_0 sequent α-regular, if it has the form $\Gamma, \Phi \xrightarrow{k} \ulcorner G \urcorner$, where Γ, Φ is a k-regular hypothesis, and G is an SL goal.

A DTL_0 sequent is called β-regular if it has the form $\Gamma \xrightarrow{\ell} k$ says P, where Γ is a 0-regular hypothesis.

Next we define an inverse translation $\llcorner\cdot\lrcorner$ from regular hypothesis to hypothesis and assertions of SL.

Definition H.5 (Inverse Translation)**.** If Γ is 0-regular then we define the *SL hypothesis* $\llcorner\Gamma\lrcorner$ as follows:

$$\llcorner\Gamma\lrcorner \;=\; \{k : \{A \mid \ell \text{ } \mathsf{claims} \text{ } k \text{ } \mathsf{says} \text{ } \ulcorner A \urcorner \in \Gamma\} \mid k \in \mathrm{SL}\}$$

Similarly, if Γ, Φ is k-regular, we define the *SL assertion* $\llcorner\Gamma, \Phi\lrcorner_k$ as follows:

$$\llcorner\Gamma, \Phi\lrcorner_k \;=\; \{A \mid \ell \text{ } \mathsf{claims} \text{ } k \text{ } \mathsf{says} \text{ } \ulcorner A \urcorner \in \Gamma\} \cup \{P \mid P \in \Phi\}$$

We now prove completeness of the translation.

Lemma H.6 (Completeness of Translation)**.** *The following hold:*

1. *(α-regular) If $\Gamma, \Phi \xrightarrow{k} \ulcorner G \urcorner$ is α-regular and provable in DTL_0, then $\llcorner\Gamma, \Phi\lrcorner_k \vdash_{\llcorner\Gamma\lrcorner} G$ in SL.*
2. *(β-regular) If $\Gamma \xrightarrow{\ell} k$ says P is β-regular and provable in DTL_0, then $\cdot \vdash_{\llcorner\Gamma\lrcorner} k$ says P in SL.*

Proof. We prove both statements simultaneously by induction on the *depths* of the given derivations in DTL_0. We remind the reader that A denotes a clause from SL, not a DTL_0 formula. We analyze cases on the last rule in the given derivation.

Proof of (1).

Case. $$\frac{P \text{ atomic}}{\Gamma, (\Phi, P) \xrightarrow{k} P}\text{init}$$

Observe that by definition, $P \in \llcorner \Gamma, (\Phi, P) \lrcorner_k$.

1. $\llcorner \Gamma, (\Phi, P) \lrcorner_k \vdash_{\llcorner \Gamma \lrcorner} P$ (Rule (bc))

Case. $$\frac{(\Gamma, k \text{ claims } \ulcorner A \urcorner), (\Phi, \ulcorner A \urcorner) \xrightarrow{k} \ulcorner G \urcorner \qquad k \succeq k}{(\Gamma, k \text{ claims } \ulcorner A \urcorner), \Phi \xrightarrow{k} \ulcorner G \urcorner}\text{claims}$$

Note that the premise is α-regular.

1. $\llcorner (\Gamma, k \text{ claims } \ulcorner A \urcorner), (\Phi, \ulcorner A \urcorner) \lrcorner_k \vdash_{\llcorner \Gamma, k \text{claims} \ulcorner A \urcorner \lrcorner} G$ (i.h. on premise)
2. $\llcorner (\Gamma, k \text{ claims } \ulcorner A \urcorner), (\Phi, \ulcorner A \urcorner) \lrcorner_k = \llcorner (\Gamma, k \text{ claims } \ulcorner A \urcorner), \Phi \lrcorner_k$ (Defn.)
3. $\llcorner (\Gamma, k \text{ claims } \ulcorner A \urcorner), \Phi \lrcorner_k \vdash_{\llcorner \Gamma, k \text{claims} \ulcorner A \urcorner \lrcorner} G$ (1, 2)

Case. $$\frac{(\Gamma, \ell \text{ claims } k' \text{ says } \ulcorner A \urcorner), k' \text{ says } \ulcorner A \urcorner, \Phi \xrightarrow{k} \ulcorner G \urcorner \qquad \ell \succeq k}{(\Gamma, \ell \text{ claims } k' \text{ says } \ulcorner A \urcorner), \Phi \xrightarrow{k} \ulcorner G \urcorner}\text{claims}$$

1. $(\Gamma, \ell \text{ claims } k' \text{ says } \ulcorner A \urcorner, k' \text{ claims } \ulcorner A \urcorner), \Phi \xrightarrow{k} \ulcorner G \urcorner$ (Lemma G.5.1 on premise)
2. $\llcorner (\Gamma, \ell \text{ claims } k' \text{ says } \ulcorner A \urcorner, k' \text{ claims } \ulcorner A \urcorner), \Phi \lrcorner_k \vdash_{\llcorner \Gamma, \ell \text{claims} k' \text{says} \ulcorner A \urcorner, k' \text{claims} \ulcorner A \urcorner \lrcorner} G$

   (i.h. on 1; the sequent in 1 is α-regular)
3. $\llcorner (\Gamma, \ell \text{ claims } k' \text{ says } \ulcorner A \urcorner, k' \text{ claims } \ulcorner A \urcorner), \Phi \lrcorner_k = \llcorner (\Gamma, \ell \text{ claims } k' \text{ says } \ulcorner A \urcorner), \Phi \lrcorner_k$

   (Defn.)
4. $\llcorner \Gamma, \ell \text{ claims } k' \text{ says } \ulcorner A \urcorner, k' \text{ claims } \ulcorner A \urcorner \lrcorner = \llcorner \Gamma, \ell \text{ claims } k' \text{ says } \ulcorner A \urcorner \lrcorner$ (Defn.)
5. $\llcorner (\Gamma, \ell \text{ claims } k' \text{ says } \ulcorner A \urcorner), \Phi \lrcorner_k \vdash_{\llcorner \Gamma, \ell \text{claims} k' \text{says} \ulcorner A \urcorner \lrcorner} G$ (2, 3, 4)

Case. $$\frac{\Gamma|_\ell \xrightarrow{\ell} k' \text{ says } P}{\Gamma, \Phi \xrightarrow{k} \ell \text{ says } k' \text{ says } P}\text{saysR}$$

Note that the premise is β-regular.

1. $\cdot \vdash_{\llcorner \Gamma|_\ell \lrcorner} k' \text{ says } P$ (i.h. (2) on the premise)
2. $\llcorner \Gamma|_\ell \lrcorner = \llcorner \Gamma \lrcorner$ (Defn.)
3. $\cdot \vdash_{\llcorner \Gamma \lrcorner} k' \text{ says } P$ (1, 2)

4. $\llcorner\Gamma, \Phi\lrcorner_k \vdash_{\llcorner\Gamma\lrcorner} k'$ says P (Lemma H.1.1 on 3)

Case.
$$\frac{\Gamma, (\Phi, (\ulcorner G_1 \urcorner, \ldots, \ulcorner G_n \urcorner) \supset P) \xrightarrow{k} \ulcorner G_1 \urcorner \wedge \ldots \wedge \ulcorner G_n \urcorner \qquad \Gamma, (\Phi, (\ulcorner G_1 \urcorner, \ldots, \ulcorner G_n \urcorner) \supset P, P) \xrightarrow{k} \ulcorner G \urcorner}{\Gamma, (\Phi, (\ulcorner G_1 \urcorner, \ldots, \ulcorner G_n \urcorner) \supset P) \xrightarrow{k} \ulcorner G \urcorner} \supset\mathrm{L}$$

1. $\Gamma, (\Phi, (\ulcorner G_1 \urcorner, \ldots, \ulcorner G_n \urcorner) \supset P) \xrightarrow{k} \ulcorner G_i \urcorner$ (Lemma G.5.2 on 1st premise)
2. $\llcorner\Gamma, (\Phi, (\ulcorner G_1 \urcorner, \ldots, \ulcorner G_n \urcorner) \supset P)\lrcorner_k \vdash_{\llcorner\Gamma\lrcorner} G_i$ (i.h. on 1)
3. k claims $(\ulcorner G_1 \urcorner, \ldots, \ulcorner G_n \urcorner) \supset P \in \Gamma$ (Defn. of k-regular)
4. ℓ claims k says $((\ulcorner G_1 \urcorner, \ldots, \ulcorner G_n \urcorner) \supset P) \in \Gamma$ (Defn. of 0-regular and 3)
5. $P \leftarrow G_1, \ldots, G_n \in \llcorner\Gamma, (\Phi, (\ulcorner G_1 \urcorner, \ldots, \ulcorner G_n \urcorner) \supset P)\lrcorner_k$ (Defn. of $\llcorner\cdot\lrcorner_k$ and 4)
6. $\llcorner\Gamma, (\Phi, (\ulcorner G_1 \urcorner, \ldots, \ulcorner G_n \urcorner) \supset P)\lrcorner_k \vdash_{\llcorner\Gamma\lrcorner} P$ (Rule (bc) on 5 and 2)
7. $\llcorner\Gamma, (\Phi, (\ulcorner G_1 \urcorner, \ldots, \ulcorner G_n \urcorner) \supset P)\lrcorner_k, P \vdash_{\llcorner\Gamma\lrcorner} G$ (i.h. on 2nd premise)
8. $\llcorner\Gamma, (\Phi, (\ulcorner G_1 \urcorner, \ldots, \ulcorner G_n \urcorner) \supset P)\lrcorner_k \vdash_{\llcorner\Gamma\lrcorner} G$ (Lemma H.1.2 on 6 and 7)

Proof of (2).

Case.
$$\frac{\Gamma, \ell \text{ claims } k' \text{ says } \ulcorner A \urcorner, k' \text{ says } \ulcorner A \urcorner \xrightarrow{\ell} k \text{ says } P \qquad \ell \succeq \ell}{\Gamma, \ell \text{ claims } k' \text{ says } \ulcorner A \urcorner \xrightarrow{\ell} k \text{ says } P} \text{claims}$$

1. Γ, ℓ claims k' says $\ulcorner A \urcorner, k'$ claims $\ulcorner A \urcorner \xrightarrow{\ell} k$ says P (Lemma G.5.1 on premise)
2. $\cdot \vdash_{\llcorner\Gamma, \ell \text{claims} k' \text{says} \ulcorner A \urcorner, k' \text{claims} \ulcorner A \urcorner\lrcorner} k$ says P (i.h. on 1)
3. $\llcorner\Gamma, \ell$ claims k' says $\ulcorner A \urcorner, k'$ claims $\ulcorner A \urcorner\lrcorner = \llcorner\Gamma, \ell$ claims k' says $\ulcorner A \urcorner\lrcorner$ (Defn.)
4. $\cdot \vdash_{\llcorner\Gamma, \ell \text{claims} k' \text{says} \ulcorner A \urcorner\lrcorner} k$ says P (2, 3)

Case.
$$\frac{\Gamma|_k \xrightarrow{k} P}{\Gamma \xrightarrow{\ell} k \text{ says } P} \text{saysR}$$

Note that the premise is α-regular (with $\Phi = \cdot$).

1. $\llcorner\Gamma|_k\lrcorner_k \vdash_{\llcorner\Gamma|_k\lrcorner} P$ (i.h. (1) on premise)
2. $\llcorner\Gamma|_k\lrcorner = \llcorner\Gamma\lrcorner$ (Defn.)
3. $\llcorner\Gamma|_k\lrcorner_k \vdash_{\llcorner\Gamma\lrcorner} P$ (1, 2)
4. $k : \llcorner\Gamma|_k\lrcorner_k \in \llcorner\Gamma\lrcorner$ (Defn.)
5. $\cdot \vdash_{\llcorner\Gamma\lrcorner} K$ says P (Rule (says) on 4 and 3)

□

Theorem H.7 (Correctness; Theorem 5.5)**.** *Suppose* $(K : \Delta) \in \Gamma$*. Then* $\Delta \vdash_\Gamma G$ *in SL if and only if* $\ulcorner\Gamma\urcorner, \ulcorner\Delta\urcorner \xrightarrow{K} \ulcorner G\urcorner$ *in DTL*$_0$*.*

Proof. Suppose $(K : \Delta) \in \Gamma$ and $\Delta \vdash_\Gamma G$ in SL. Then $\ulcorner\Gamma\urcorner, \ulcorner\Delta\urcorner \xrightarrow{K} \ulcorner G\urcorner$ by Lemma H.2.

Conversely, suppose that $(K : \Delta) \in \Gamma$ and $\ulcorner\Gamma\urcorner, \ulcorner\Delta\urcorner \xrightarrow{K} \ulcorner G\urcorner$ in DTL$_0$. Let $\Gamma = (K_i : \Delta_i)_{i\in\{1,\ldots,m\}}$, where each $\Delta_i = A_{1,i}, \ldots, A_{n_i,i}$. Then $\ulcorner\Gamma\urcorner = (\ell \text{ says } K_i \text{ says } \ulcorner A_{1,i}\urcorner, \ldots, \ell \text{ says } K_i \text{ says } \ulcorner A_{n_i,i}\urcorner)_{i\in\{1,\ldots,m\}}$. Further assume that $K = K_t$ (where $t \in \{1, \ldots, m\}$), so that $\Delta = A_{1,t}, \ldots, A_{n_t,t}$. Then we have,

1. $(\ell \text{ says } K_i \text{ says } \ulcorner A_{1,i}\urcorner, \ldots, \ell \text{ says } K_i \text{ says } \ulcorner A_{n_i,}\urcorner)_{i\in\{1,\ldots,m\}}, \ulcorner A_{1,t}\urcorner, \ldots, \ulcorner A_{n_t,t}\urcorner \xrightarrow{K_t} \ulcorner G\urcorner$ (Assumption)

2. $(\ell \text{ claims } K_i \text{ says } \ulcorner A_{1,i}\urcorner, \ldots, \ell \text{ claims } K_i \text{ says } \ulcorner A_{n_i,i}\urcorner)_{i\in\{1,\ldots,m\}}, \ulcorner A_{1,t}\urcorner, \ldots, \ulcorner A_{n_t,t}\urcorner \xrightarrow{K_t} \ulcorner G\urcorner$ (Lemma G.5.1)

3. $(\ell \text{ claims } K_i \text{ says } \ulcorner A_{1,i}\urcorner, \ldots, \ell \text{ claims } K_i \text{ says } \ulcorner A_{n_i,i}\urcorner)_{i\in\{1,\ldots,m\}}, K_t \text{ claims } \ulcorner A_{1,t}\urcorner, \ldots,$ $K_t \text{ claims } \ulcorner A_{n_t,t}\urcorner, \ulcorner A_{1,t}\urcorner, \ldots, \ulcorner A_{n_t,t}\urcorner \xrightarrow{K_t} \ulcorner G\urcorner$ (Weakening)

4. $\Gamma = \llcorner(\ell \text{ claims } K_i \text{ says } \ulcorner A_{1,i}\urcorner, \ldots, \ell \text{ claims } K_i \text{ says } \ulcorner A_{n_i,i}\urcorner)_{i\in\{1,\ldots,m\}}, K_t \text{ claims } \ulcorner A_{1,t}\urcorner, \ldots, K_t \text{ claims } \ulcorner A_{n_t,t}\urcorner\lrcorner$ (Defn.)

5. $\Delta = \llcorner(\ell \text{ claims } K_i \text{ says } \ulcorner A_{1,i}\urcorner, \ldots, \ell \text{ claims } K_i \text{ says } \ulcorner A_{n_i,i}\urcorner)_{i\in\{1,\ldots,m\}}, K_t \text{ claims } \ulcorner A_{1,t}\urcorner, \ldots, K_t \text{ claims } \ulcorner A_{n_t,t}\urcorner, \ulcorner A_{1,t}\urcorner, \ldots, \ulcorner A_{n_t,t}\urcorner\lrcorner_{K_t}$ (Defn.)

6. $\Delta \vdash_\Gamma G$

   (Lemma H.6.1 on 3 using 4 and 5 to abbreviate; the sequent in 3 is α-regular)

□

I Proofs from Section 5.5

In this appendix we prove the Theorems related to BL$_0$ (Section 5.5). Many of these theorems rely on the Hilbert style axiomatization for BL$_0$, which we develop first.

I.1 The Axiomatic System for BL$_0$

In Section 5.5, we presented some rules and axioms for the axiomatic system of BL$_0$. Here, we list all the rules and axioms, including those listed earlier.

$$\frac{\vdash A}{\vdash K \text{ says } A}\text{nec} \qquad \frac{\vdash A \supset B \quad \vdash A}{\vdash B}\text{mp} \qquad \frac{A \text{ is an axiom}}{\vdash A}\text{ax}$$

Axioms:

$$
\begin{array}{ll}
(K \textsf{ says } (A \supset B)) \supset ((K \textsf{ says } A) \supset (K \textsf{ says } B)) & \text{(K)} \\
(K \textsf{ says } A) \supset K' \textsf{ says } K \textsf{ says } A & \text{(Bind)} \\
K \textsf{ says } ((K \textsf{ says } A) \supset A) & \text{(C)} \\
A \supset (B \supset A) & \text{(imp1)} \\
(A \supset B) \supset ((A \supset (B \supset C)) \supset (A \supset C)) & \text{(imp2)} \\
A \supset (B \supset (A \wedge B)) & \text{(conj1)} \\
(A \wedge B) \supset A & \text{(conj2)} \\
(A \wedge B) \supset B & \text{(conj3)} \\
A \supset (A \vee B) & \text{(disj1)} \\
B \supset (A \vee B) & \text{(disj2)} \\
(A \supset C) \supset ((B \supset C) \supset ((A \vee B) \supset C)) & \text{(disj3)} \\
\top & \text{(true)} \\
\bot \supset A & \text{(false)}
\end{array}
$$

Next, as we did for DTL_0 in Appendix C, we introduce a *generalized* axiomatic system for BL_0. Let Γ denote a multi set of formulas (not judgments). We write $\Gamma \vdash_G A$ to mean that A may be established from assumptions Γ. The rules of the generalized axiomatic system are:

$$
\frac{}{\Gamma, A \vdash_G A}\text{use} \qquad \frac{\cdot \vdash_G A}{\Gamma \vdash_G K \textsf{ says } A}\text{nec} \qquad \frac{\Gamma \vdash_G A \supset B \qquad \Gamma \vdash_G A}{\Gamma \vdash_G B}\text{mp}
$$

$$
\frac{A \text{ is an axiom}}{\Gamma \vdash_G A}\text{ax}
$$

Now we prove some basic properties of the generalized axiomatic system, including the deduction theorem, and show that the generalized system reduces to the axiomatic system when Γ is empty.

Lemma I.1 (Basic properties)**.** *The following hold.*

1. *(Weakening)* $\Gamma \vdash_G A$ *implies* $\Gamma, \Gamma' \vdash_G A$
2. *(Substitution)* $\Gamma \vdash_G A$ *and* $\Gamma, A \vdash_G B$ *imply* $\Gamma \vdash_G B$

Proof. Exactly as for DTL_0 in Lemma C.1. The proof does not rely on the axiom (4), which is the only difference between the two systems. □

Theorem I.2 (Deduction)**.** *The following hold.*

1. $\Gamma \vdash_G A \supset B$ *implies* $\Gamma, A \vdash_G B$
2. $\Gamma, A \vdash_G B$ *implies* $\Gamma \vdash_G A \supset B$

Proof. Exactly as for DTL_0 in Theorem C.2. The proof does not rely on the axiom (4), which is the only difference between the two systems. □

Theorem I.3 (G iff Axiomatic)**.** $\vdash A$ *if and only if* $\cdot \vdash_G A$

Proof. In each direction by straightforward induction on the given derivation. □

I.2 Proofs of Theorems 5.6 and 5.7

We simultaneously prove Theorem 5.6 (Equivalence of sequent calculus and axiomatic system for BL_0), and Theorem 5.7 (Correctness of translation from BL_0 to DTL_0). To do this we establish three lemmas.

Lemma I.4 (Sequent Calculus ⇒ Axiomatic System)**.** *$\Gamma \xrightarrow{K} A$ in BL_0's sequent calculus (Figure 5) implies $\cdot \vdash_G K\ \mathsf{says}\ (\overline{\Gamma} \supset A)$ in BL_0's generalized axiomatic system.*

Proof. We induct on the given derivation of $\Gamma \xrightarrow{K} A$, and show some cases related to claims and says. We freely use properties such as Currying in the axiomatic system.

Case. $\dfrac{P \text{ atomic}}{\Gamma, P \xrightarrow{K} P}\text{init}$

1. $\cdot \vdash_G (\overline{\Gamma} \wedge P) \supset P$ (Rule (ax) and (conj3))
2. $\cdot \vdash_G K\ \mathsf{says}\ ((\overline{\Gamma} \wedge P) \supset P)$ (Rule (nec))

Case. $\dfrac{\Gamma, K\ \mathsf{claims}\ A, A \xrightarrow{K} C}{\Gamma, K\ \mathsf{claims}\ A \xrightarrow{K} C}\text{claims}$

1. $\cdot \vdash_G K\ \mathsf{says}\ ((\overline{\Gamma} \wedge (K\ \mathsf{says}\ A) \wedge A) \supset C)$ (i.h. on premise)
2. $\cdot \vdash_G ((\overline{\Gamma} \wedge K\ \mathsf{says}\ A) \supset A) \supset (((\overline{\Gamma} \wedge K\ \mathsf{says}\ A) \supset (A \supset (\overline{\Gamma} \wedge (K\ \mathsf{says}\ A) \wedge A))) \supset ((\overline{\Gamma} \wedge K\ \mathsf{says}\ A) \supset (\overline{\Gamma} \wedge (K\ \mathsf{says}\ A) \wedge A)))$ (Rule (ax) and (imp2))
3. $\cdot \vdash_G (K\ \mathsf{says}\ ((\overline{\Gamma} \wedge K\ \mathsf{says}\ A) \supset A)) \supset ((K\ \mathsf{says}\ ((\overline{\Gamma} \wedge K\ \mathsf{says}\ A) \supset (A \supset (\overline{\Gamma} \wedge (K\ \mathsf{says}\ A) \wedge A)))) \supset K\ \mathsf{says}\ ((\overline{\Gamma} \wedge K\ \mathsf{says}\ A) \supset (\overline{\Gamma} \wedge (K\ \mathsf{says}\ A) \wedge A)))$

   (Rule (ax), (K), Rule (mp))
4. $\cdot \vdash_G K\ \mathsf{says}\ ((\overline{\Gamma} \wedge K\ \mathsf{says}\ A) \supset A)$ (Theorem in G, follows from (C))
5. $\cdot \vdash_G (K\ \mathsf{says}\ ((\overline{\Gamma} \wedge K\ \mathsf{says}\ A) \supset (A \supset (\overline{\Gamma} \wedge (K\ \mathsf{says}\ A) \wedge A)))) \supset K\ \mathsf{says}\ ((\overline{\Gamma} \wedge K\ \mathsf{says}\ A) \supset (\overline{\Gamma} \wedge (K\ \mathsf{says}\ A) \wedge A))$ (Rule (mp) on 3 and 4)
6. $\cdot \vdash_G (\overline{\Gamma} \wedge K\ \mathsf{says}\ A) \supset (A \supset (\overline{\Gamma} \wedge (K\ \mathsf{says}\ A) \wedge A))$ (Currying)
7. $\cdot \vdash_G K\ \mathsf{says}\ ((\overline{\Gamma} \wedge K\ \mathsf{says}\ A) \supset (A \supset (\overline{\Gamma} \wedge (K\ \mathsf{says}\ A) \wedge A)))$ (Rule (nec))
8. $\cdot \vdash_G K\ \mathsf{says}\ ((\overline{\Gamma} \wedge K\ \mathsf{says}\ A) \supset (\overline{\Gamma} \wedge (K\ \mathsf{says}\ A) \wedge A))$ (Rule (mp) on 5 and 7)
9. $\cdot \vdash_G (A \supset B) \supset ((B \supset C) \supset (A \supset C))$ (Theorem in G)
10. $\cdot \vdash_G K\ \mathsf{says}\ ((A \supset B) \supset ((B \supset C) \supset (A \supset C)))$ (Rule (nec))
11. $\cdot \vdash_G (K\ \mathsf{says}\ (A \supset B)) \supset ((K\ \mathsf{says}\ (B \supset C)) \supset K\ \mathsf{says}\ (A \supset C))$

    (Rule (ax), K, rule (mp))

12. $\cdot \vdash_G (K \text{ says } ((\overline{\Gamma} \wedge K \text{ says } A) \supset (\overline{\Gamma} \wedge (K \text{ says } A) \wedge A))) \supset ((K \text{ says } ((\overline{\Gamma} \wedge (K \text{ says } A) \wedge A) \supset C)) \supset K \text{ says } ((\overline{\Gamma} \wedge K \text{ says } A) \supset C))$ (Instantiate 11)

13. $\cdot \vdash_G (K \text{ says } ((\overline{\Gamma} \wedge (K \text{ says } A) \wedge A) \supset C)) \supset K \text{ says } ((\overline{\Gamma} \wedge K \text{ says } A) \supset C)$

    (Rule (mp) on 12 and 8)

14. $\cdot \vdash_G K \text{ says } ((\overline{\Gamma} \wedge K \text{ says } A) \supset C)$ (Rule (mp) on 13 and 1)

Case. $$\frac{\Gamma| \xrightarrow{K} A}{\Gamma \xrightarrow{K'} K \text{ says } A}\text{saysR}$$

Let $\Gamma| = K_1 \text{ says } A_1, \ldots, K_n \text{ says } A_n$

1. $\cdot \vdash_G K \text{ says } ((K_1 \text{ says } A_1 \wedge \ldots \wedge K_n \text{ says } A_n) \supset A)$ (i.h. on premise)
2. $\cdot \vdash_G (K \text{ says } K_1 \text{ says } A_1 \wedge \ldots \wedge K \text{ says } K_n \text{ says } A_n) \supset K \text{ says } A$ (Rule (ax), K, and Rule (mp))
3. $K \text{ says } K_1 \text{ says } A_1, \ldots, K \text{ says } K_n \text{ says } A_n \vdash_G K \text{ says } A$ (Theorem I.2)
4. $\cdot \vdash_G (K_i \text{ says } A_i) \supset (K \text{ says } K_i \text{ says } A_i)$ (Rule (ax) and (Bind))
5. $K_i \text{ says } A_i \vdash_G K \text{ says } K_i \text{ says } A_i$ (Theorem I.2)
6. $K_1 \text{ says } A_1, \ldots, K_n \text{ says } A_n \vdash_G K \text{ says } A$ (Lemma I.1.2 on 5 and 3)
7. $\Gamma \vdash_G K \text{ says } A$ (Lemma I.1.1)
8. $\cdot \vdash_G \overline{\Gamma} \supset K \text{ says } A$ (Theorem I.2)
9. $\cdot \vdash_G K' \text{ says } (\overline{\Gamma} \supset K \text{ says } A)$ (Rule (nec))

Case. $$\frac{\Gamma, K \text{ says } A, K \text{ claims } A \xrightarrow{K'} C}{\Gamma, K \text{ says } A \xrightarrow{K'} C}\text{saysL}$$

1. $\cdot \vdash_G K' \text{ says } ((\overline{\Gamma} \wedge (K \text{ says } A) \wedge (K \text{ says } A)) \supset C)$ (i.h. on premise)
2. $\cdot \vdash_G (\overline{\Gamma} \wedge K \text{ says } A) \supset (\overline{\Gamma} \wedge (K \text{ says } A) \wedge (K \text{ says } A))$ (Theorem in G)
3. $\cdot \vdash_G K' \text{ says } ((\overline{\Gamma} \wedge K \text{ says } A) \supset (\overline{\Gamma} \wedge (K \text{ says } A) \wedge (K \text{ says } A)))$ (Rule (nec))
4. $\cdot \vdash_G (A \supset B) \supset ((B \supset C) \supset (A \supset C))$ (Theorem in G)
5. $\cdot \vdash_G K' \text{ says } ((A \supset B) \supset ((B \supset C) \supset (A \supset C)))$ (Rule (nec))
6. $\cdot \vdash_G (K' \text{ says } (A \supset B)) \supset ((K' \text{ says } (B \supset C)) \supset K' \text{ says } (A \supset C))$

   (Rule (ax), K, rule (mp))
7. $\cdot \vdash_G (K' \text{ says } ((\overline{\Gamma} \wedge K \text{ says } A) \supset (\overline{\Gamma} \wedge (K \text{ says } A) \wedge (K \text{ says } A)))) \supset ((K' \text{ says } ((\overline{\Gamma} \wedge (K \text{ says } A) \wedge (K \text{ says } A)) \supset C)) \supset K' \text{ says } ((\overline{\Gamma} \wedge K \text{ says } A) \supset C))$

(Instantiate 6)

8. $\cdot \vdash_G (K' \text{ says } ((\overline{\Gamma} \wedge (K \text{ says } A) \wedge (K \text{ says } A)) \supset C)) \supset K' \text{ says } ((\overline{\Gamma} \wedge K \text{ says } A) \supset C)$ (Rule (mp) on 7 and 3)

9. $\cdot \vdash_G K' \text{ says } ((\overline{\Gamma} \wedge K \text{ says } A) \supset C)$ (Rule (mp) on 8 and 1)

□

Lemma I.5 (Soundness of Translation)**.** *If* $\vdash A$ *in* BL_0*, then for every* K*,* $\cdot \xrightarrow{K} [\![A]\!]$ *in* DTL_0*.*

Proof. We induct on the derivation of $\vdash A$, and case analyze the last rule.

Case. $\dfrac{\vdash A}{\vdash K' \text{ says } A}\text{nec}$

To show: $\cdot \xrightarrow{K} \ell \text{ says } K' \text{ says } [\![A]\!]$.

1. $\cdot \xrightarrow{K'} [\![A]\!]$ (i.h. on premise)
2. $\cdot \xrightarrow{\ell} K' \text{ says } [\![A]\!]$ (Rule (saysR))
3. $\cdot \xrightarrow{K} \ell \text{ says } K' \text{ says } [\![A]\!]$ (Rule (saysR))

Case. $\dfrac{\vdash A \supset B \qquad \vdash A}{\vdash B}\text{mp}$

1. $\cdot \xrightarrow{K} [\![A]\!] \supset [\![B]\!]$ (i.h. on 1st premise)
2. $\cdot \xrightarrow{K} [\![A]\!]$ (i.h. on 2nd premise)
3. $[\![A]\!], [\![A]\!] \supset [\![B]\!] \xrightarrow{K} [\![B]\!]$ (Theorem in DTL_0)
4. $[\![A]\!] \supset [\![B]\!] \xrightarrow{K} [\![B]\!]$ (Theorem B.2 on 3 and 2)
5. $\cdot \xrightarrow{K} [\![B]\!]$ (Theorem B.2 on 4 and 1)

Case. $\dfrac{A \text{ is an axiom}}{\vdash A}\text{ax}$

To show: $\cdot \xrightarrow{K} [\![A]\!]$. We case analyze all the axioms, showing only some of the important cases here. The other cases are straightforward.

Case. (Axiom K) $A = (K' \text{ says } (A' \supset B')) \supset ((K' \text{ says } A') \supset (K' \text{ says } B'))$

To show: $\cdot \xrightarrow{K} (\ell \text{ says } K' \text{ says } ([\![A']\!] \supset [\![B']\!])) \supset ((\ell \text{ says } K' \text{ says } [\![A']\!]) \supset (\ell \text{ says } K' \text{ says } [\![B']\!]))$

1. $[\![A']\!] \supset [\![B']\!], [\![A']\!] \xrightarrow{K'} [\![A']\!]$ (Theorem B.3)

2. $[\![A']\!] \supset [\![B']\!], [\![A']\!], [\![B']\!] \xrightarrow{K'} [\![B']\!]$ (Theorem B.3)

3. $[\![A']\!] \supset [\![B']\!], [\![A']\!] \xrightarrow{K'} [\![B']\!]$ (Rule ($\supset$L))

4. $K'\ \mathsf{claims}\ ([\![A']\!] \supset [\![B']\!]), [\![A']\!] \supset [\![B']\!], K'\ \mathsf{claims}\ [\![A']\!], [\![A']\!] \xrightarrow{K'} [\![B']\!]$ (Weakening)

5. $K'\ \mathsf{claims}\ ([\![A']\!] \supset [\![B']\!]), K'\ \mathsf{claims}\ [\![A']\!] \xrightarrow{K'} [\![B']\!]$ (Rule (claims) twice)

6. $K'\ \mathsf{says}\ ([\![A']\!] \supset [\![B']\!]), K'\ \mathsf{claims}\ ([\![A']\!] \supset [\![B']\!]), K'\ \mathsf{says}\ [\![A']\!], K'\ \mathsf{claims}\ [\![A']\!] \xrightarrow{\ell} K'\ \mathsf{says}\ [\![B']\!]$ (Rule (saysR))

7. $K'\ \mathsf{says}\ ([\![A']\!] \supset [\![B']\!]), K'\ \mathsf{says}\ [\![A']\!] \xrightarrow{\ell} K'\ \mathsf{says}\ [\![B']\!]$ (Rule (saysL) twice)

8. $\ell\ \mathsf{claims}\ K'\ \mathsf{says}\ ([\![A']\!] \supset [\![B']\!]), K'\ \mathsf{says}\ ([\![A']\!] \supset [\![B']\!]), \ell\ \mathsf{claims}\ K'\ \mathsf{says}\ [\![A']\!], K'\ \mathsf{says}\ [\![A']\!] \xrightarrow{\ell} K'\ \mathsf{says}\ [\![B']\!]$ (Weakening)

9. $\ell\ \mathsf{claims}\ K'\ \mathsf{says}\ ([\![A']\!] \supset [\![B']\!]), \ell\ \mathsf{claims}\ K'\ \mathsf{says}\ [\![A']\!] \xrightarrow{\ell} K'\ \mathsf{says}\ [\![B']\!]$ (Rule (claims) twice)

10. $\ell\ \mathsf{claims}\ K'\ \mathsf{says}\ ([\![A']\!] \supset [\![B']\!]), \ell\ \mathsf{says}\ K'\ \mathsf{says}\ ([\![A']\!] \supset [\![B']\!]), \ell\ \mathsf{claims}\ K'\ \mathsf{says}\ [\![A']\!], \ell\ \mathsf{says}\ K'\ \mathsf{says}\ [\![A']\!] \xrightarrow{K} \ell\ \mathsf{says}\ K'\ \mathsf{says}\ [\![B']\!]$ (Rule (saysR))

11. $\ell\ \mathsf{says}\ K'\ \mathsf{says}\ ([\![A']\!] \supset [\![B']\!]), \ell\ \mathsf{says}\ K'\ \mathsf{says}\ [\![A']\!] \xrightarrow{K} \ell\ \mathsf{says}\ K'\ \mathsf{says}\ [\![B']\!]$ (Rule (saysL) twice)

12. $\cdot \xrightarrow{K} (\ell\ \mathsf{says}\ K'\ \mathsf{says}\ ([\![A']\!] \supset [\![B']\!])) \supset ((\ell\ \mathsf{says}\ K'\ \mathsf{says}\ [\![A']\!]) \supset (\ell\ \mathsf{says}\ K'\ \mathsf{says}\ [\![B']\!]))$ (Rule ($\supset$R) twice)

Case. (Axiom Bind) $A = (K'\ \mathsf{says}\ A') \supset K''\ \mathsf{says}\ K'\ \mathsf{says}\ A'$

To show: $\cdot \xrightarrow{K} (\ell\ \mathsf{says}\ K'\ \mathsf{says}\ [\![A']\!]) \supset \ell\ \mathsf{says}\ K''\ \mathsf{says}\ \ell\ \mathsf{says}\ K'\ \mathsf{says}\ [\![A']\!]$

1. $\ell\ \mathsf{claims}\ K'\ \mathsf{says}\ [\![A']\!], K'\ \mathsf{says}\ [\![A']\!], K'\ \mathsf{claims}\ [\![A']\!], [\![A']\!] \xrightarrow{K'} [\![A']\!]$ (Theorem B.3)

2. $\ell\ \mathsf{claims}\ K'\ \mathsf{says}\ [\![A']\!], K'\ \mathsf{says}\ [\![A']\!], K'\ \mathsf{claims}\ [\![A']\!] \xrightarrow{K'} [\![A']\!]$ (Rule (claims))

3. $\ell\ \mathsf{claims}\ K'\ \mathsf{says}\ [\![A']\!], K'\ \mathsf{says}\ [\![A']\!] \xrightarrow{K'} [\![A']\!]$ (Rule (saysL))

4. $\ell\ \mathsf{claims}\ K'\ \mathsf{says}\ [\![A']\!] \xrightarrow{K'} [\![A']\!]$ (Rule (claims))

5. $\ell\ \mathsf{claims}\ K'\ \mathsf{says}\ [\![A']\!] \xrightarrow{\ell} K'\ \mathsf{says}\ [\![A']\!]$ (Rule (saysR))

6. $\ell\ \mathsf{claims}\ K'\ \mathsf{says}\ [\![A']\!] \xrightarrow{K''} \ell\ \mathsf{says}\ K'\ \mathsf{says}\ [\![A']\!]$ (Rule (saysR))

7. $\ell\ \mathsf{claims}\ K'\ \mathsf{says}\ [\![A']\!] \xrightarrow{\ell} K''\ \mathsf{says}\ \ell\ \mathsf{says}\ K'\ \mathsf{says}\ [\![A']\!]$ (Rule (saysR))

8. $\ell\ \mathsf{claims}\ K'\ \mathsf{says}\ [\![A']\!], \ell\ \mathsf{says}\ K'\ \mathsf{says}\ [\![A']\!] \xrightarrow{K} \ell\ \mathsf{says}\ K''\ \mathsf{says}\ \ell\ \mathsf{says}\ K'\ \mathsf{says}\ [\![A']\!]$ (Rule (saysR))

9. $\ell\ \mathsf{says}\ K'\ \mathsf{says}\ [\![A']\!] \xrightarrow{K} \ell\ \mathsf{says}\ K''\ \mathsf{says}\ \ell\ \mathsf{says}\ K'\ \mathsf{says}\ [\![A']\!]$ (Rule (saysL))

10. $\cdot \xrightarrow{K} (\ell\ \mathsf{says}\ K'\ \mathsf{says}\ [\![A']\!]) \supset \ell\ \mathsf{says}\ K''\ \mathsf{says}\ \ell\ \mathsf{says}\ K'\ \mathsf{says}\ [\![A']\!]$ (Rule ($\supset$R))

Case. (Axiom C) $A = K' \mathsf{~says~} ((K' \mathsf{~says~} A') \supset A')$

To show: $\cdot \xrightarrow{K} \ell \mathsf{~says~} K' \mathsf{~says~} ((\ell \mathsf{~says~} K' \mathsf{~says~} [\![A']\!]) \supset [\![A']\!])$

1. $\ell \mathsf{~claims~} K' \mathsf{~says~} [\![A']\!], K' \mathsf{~says~} [\![A']\!], K' \mathsf{~claims~} [\![A']\!], [\![A']\!] \xrightarrow{K'} [\![A']\!]$ (Theorem B.3)
2. $\ell \mathsf{~claims~} K' \mathsf{~says~} [\![A']\!], K' \mathsf{~says~} [\![A']\!], K' \mathsf{~claims~} [\![A']\!] \xrightarrow{K'} [\![A']\!]$ (Rule (claims))
3. $\ell \mathsf{~claims~} K' \mathsf{~says~} [\![A']\!], K' \mathsf{~says~} [\![A']\!] \xrightarrow{K'} [\![A']\!]$ (Rule (saysL))
4. $\ell \mathsf{~claims~} K' \mathsf{~says~} [\![A']\!] \xrightarrow{K'} [\![A']\!]$ (Rule (claims))
5. $\ell \mathsf{~says~} K' \mathsf{~says~} [\![A']\!], \ell \mathsf{~claims~} K' \mathsf{~says~} [\![A']\!] \xrightarrow{K'} [\![A']\!]$ (Weakening)
6. $\ell \mathsf{~says~} K' \mathsf{~says~} [\![A']\!] \xrightarrow{K'} [\![A']\!]$ (Rule (saysL))
7. $\cdot \xrightarrow{K'} (\ell \mathsf{~says~} K' \mathsf{~says~} [\![A']\!]) \supset [\![A']\!]$ (Rule ($\supset$R))
8. $\cdot \xrightarrow{\ell} K' \mathsf{~says~} ((\ell \mathsf{~says~} K' \mathsf{~says~} [\![A']\!]) \supset [\![A']\!])$ (Rule (saysR))
9. $\cdot \xrightarrow{K} \ell \mathsf{~says~} K' \mathsf{~says~} ((\ell \mathsf{~says~} K' \mathsf{~says~} [\![A']\!]) \supset [\![A']\!])$ (Rule (saysR))

□

Finally, we seek to show that whenever $[\![\Gamma]\!] \xrightarrow{K} [\![A]\!]$ in DTL_0, it is the case that $\Gamma \xrightarrow{K} A$ in BL_0. To do this, we syntactically characterize the sequents that may occur in a proof of $[\![\Gamma]\!] \xrightarrow{K} [\![A]\!]$. We call these sequents regular sequents. We further categorize regular sequents into two: α-regular and β-regular. As a general convention, we use the lowercase letter k to denote principals from BL_0, i.e., principals distinct from ℓ. By assumption, such principals are unrelated to each other in the order $\succeq$.

Definition I.6 (Regular Sequents)**.** A DTL_0 sequent is called regular if it has one of the two forms:

1. (α-regular) $\Gamma \xrightarrow{k} [\![A]\!]$, where Γ contains assumptions of following forms only: $[\![B]\!]$, $k \mathsf{~says~} [\![B]\!]$, $k \mathsf{~claims~} [\![B]\!]$, and $\ell \mathsf{~claims~} k \mathsf{~says~} [\![B]\!]$.

2. (β-regular) $\Gamma \xrightarrow{\ell} k \mathsf{~says~} [\![A]\!]$, where Γ contains assumptions of the following forms only: $k \mathsf{~says~} [\![B]\!]$, $k \mathsf{~claims~} [\![B]\!]$, and $\ell \mathsf{~claims~} k \mathsf{~says~} [\![B]\!]$.

Note that the difference between the hypothesis allowed in α-regular and β-regular sequents is that the former may contain assumptions of the form $[\![B]\!]$ whereas the latter may not.

Next, we define an inverse translation from hypothesis of regular sequents to BL_0. We denote the translation using the notation $\llcorner \cdot \lrcorner$.

Definition I.7 (Inverse Translation)**.** The inverse translation for regular hypothesis $\llcorner \Gamma \lrcorner$ is defined pointwise on the assumptions, where the inverse translation of assumptions is defined as follows:

$$\begin{array}{rcl} \llcorner [\![A]\!] \lrcorner & = & A \\ \llcorner k \mathsf{~says~} [\![A]\!] \lrcorner & = & k \mathsf{~claims~} A \\ \llcorner k \mathsf{~claims~} [\![A]\!] \lrcorner & = & k \mathsf{~claims~} A \\ \llcorner \ell \mathsf{~claims~} k \mathsf{~says~} [\![A]\!] \lrcorner & = & k \mathsf{~claims~} A \end{array}$$

Lemma I.8 (Completeness of Translation)**.** *The following hold.*

1. *If* $\Gamma \xrightarrow{k} [\![A]\!]$ *is α-regular and provable in DTL_0, then* $\llcorner\Gamma\lrcorner \xrightarrow{k} A$ *is provable in BL_0.*

2. *If* $\Gamma \xrightarrow{\ell} k \mathsf{\ says\ } [\![A]\!]$ *is β-regular and provable in DTL_0, then* $\llcorner\Gamma\lrcorner \xrightarrow{k} A$ *is provable in BL_0.*

Proof. We simultaneously prove the two clauses of the Lemma by induction on the derivations of the given regular sequents. For each clause, we analyze cases of the last rule in the derivation. We assume that weakening and strengthening for hypothesis holds in BL_0. These may be established easily by induction on derivations.

Proof of (1).

Case. $$\frac{P \text{ atomic}}{\Gamma, [\![P]\!] \xrightarrow{k} [\![P]\!]}\text{init}$$

1. $\llcorner\Gamma\lrcorner, P \xrightarrow{k} P$ (Rule (init))

Case. $$\frac{\Gamma, k \mathsf{\ claims\ } [\![A]\!], [\![A]\!] \xrightarrow{k} [\![C]\!] \qquad k \succeq k}{\Gamma, k \mathsf{\ claims\ } [\![A]\!] \xrightarrow{k} [\![C]\!]}\text{claims}$$

1. $\llcorner\Gamma\lrcorner, k \mathsf{\ claims\ } A, A \xrightarrow{k} C$ (i.h. on premise)
2. $\llcorner\Gamma\lrcorner, k \mathsf{\ claims\ } A \xrightarrow{k} C$ (Rule (claims))

Case. $$\frac{\Gamma, \ell \mathsf{\ claims\ } k' \mathsf{\ says\ } [\![A]\!], k' \mathsf{\ says\ } [\![A]\!] \xrightarrow{k} [\![C]\!] \qquad \ell \succeq k}{\Gamma, \ell \mathsf{\ claims\ } k' \mathsf{\ says\ } [\![A]\!] \xrightarrow{k} [\![C]\!]}\text{claims}$$

1. $\llcorner\Gamma\lrcorner, k' \mathsf{\ claims\ } A, k' \mathsf{\ claims\ } A \xrightarrow{k} C$ (i.h. on premise)
2. $\llcorner\Gamma\lrcorner, k' \mathsf{\ claims\ } A \xrightarrow{k} C$ (Strengthening)

Case. $$\frac{\Gamma|_\ell \xrightarrow{\ell} k' \mathsf{\ says\ } [\![A]\!]}{\Gamma \xrightarrow{k} \ell \mathsf{\ says\ } k' \mathsf{\ says\ } [\![A]\!]}\mathsf{saysR}$$

By definition, $\ell \mathsf{\ says\ } k' \mathsf{\ says\ } [\![A]\!] = [\![k' \mathsf{\ says\ } A]\!]$. Therefore we need to show that $\llcorner\Gamma\lrcorner \xrightarrow{k} k' \mathsf{\ says\ } A$. Assume that $\Gamma|_\ell = \ell \mathsf{\ claims\ } k_1 \mathsf{\ says\ } [\![A_1]\!], \ldots, \ell \mathsf{\ claims\ } k_n \mathsf{\ says\ } [\![A_n]\!]$. Note that $\llcorner\Gamma\lrcorner \supseteq k_1 \mathsf{\ claims\ } A_1, \ldots, k_n \mathsf{\ claims\ } A_n$.

1. $k_1 \mathsf{\ claims\ } A_1, \ldots, k_n \mathsf{\ claims\ } A_n \xrightarrow{k'} A$ (i.h. (2) on premise)
2. $k_1 \mathsf{\ claims\ } A_1, \ldots, k_n \mathsf{\ claims\ } A_n \xrightarrow{k} k' \mathsf{\ says\ } A$ (Rule (saysR))
3. $\llcorner\Gamma\lrcorner \xrightarrow{k} k' \mathsf{\ says\ } A$ (Weakening)

Note that the application of (saysR) in step 2 is allowed in BL_0, but not in DTL_0.

Case. $\dfrac{\Gamma, k' \textsf{ says } [\![A]\!], k' \textsf{ claims } [\![A]\!] \xrightarrow{k} [\![C]\!]}{\Gamma, k' \textsf{ says } [\![A]\!] \xrightarrow{k} [\![C]\!]}\textsf{saysL}$

1. $\llcorner\Gamma\lrcorner, k' \textsf{ claims } A, k' \textsf{ claims } A \xrightarrow{k} C$ (i.h. on premise)
2. $\llcorner\Gamma\lrcorner, k' \textsf{ claims } A \xrightarrow{k} C$ (Strengthening)

Case. $\dfrac{\Gamma, \ell \textsf{ says } k' \textsf{ says } [\![A]\!], \ell \textsf{ claims } k' \textsf{ says } [\![A]\!] \xrightarrow{k} [\![C]\!]}{\Gamma, \ell \textsf{ says } k' \textsf{ says } [\![A]\!] \xrightarrow{k} [\![C]\!]}\textsf{saysL}$

1. $\llcorner\Gamma\lrcorner, k' \textsf{ says } A, k' \textsf{ claims } A \xrightarrow{k} C$ (i.h. on premise)
2. $\llcorner\Gamma\lrcorner, k' \textsf{ says } A \xrightarrow{k} C$ (Rule (saysL))

Case. $\dfrac{\Gamma \xrightarrow{k} [\![A]\!] \qquad \Gamma \xrightarrow{k} [\![B]\!]}{\Gamma \xrightarrow{k} [\![A]\!] \wedge [\![B]\!]}\wedge\mathrm{R}$

1. $\llcorner\Gamma\lrcorner \xrightarrow{k} A$ (i.h. on 1st premise)
2. $\llcorner\Gamma\lrcorner \xrightarrow{k} B$ (i.h. on 2nd premise)
3. $\llcorner\Gamma\lrcorner \xrightarrow{k} A \wedge B$ (Rule ($\wedge$R))

Case. $\dfrac{\Gamma, [\![A]\!] \wedge [\![B]\!], [\![A]\!], [\![B]\!] \xrightarrow{k} [\![C]\!]}{\Gamma, [\![A]\!] \wedge [\![B]\!] \xrightarrow{k} [\![C]\!]}\wedge\mathrm{L}$

1. $\llcorner\Gamma\lrcorner, A \wedge B, A, B \xrightarrow{k} C$ (i.h. on premise)
2. $\llcorner\Gamma\lrcorner, A \wedge B \xrightarrow{k} C$ (Rule ($\wedge$L))

Case. $\dfrac{\Gamma \xrightarrow{k} [\![A]\!]}{\Gamma \xrightarrow{k} [\![A]\!] \vee [\![B]\!]}\vee \mathrm{R}_1$

1. $\llcorner\Gamma\lrcorner \xrightarrow{k} A$ (i.h. on premise)
2. $\llcorner\Gamma\lrcorner \xrightarrow{k} A \vee B$ (Rule ($\vee$ R_1))

Case. $\dfrac{\Gamma \xrightarrow{k} [\![B]\!]}{\Gamma \xrightarrow{k} [\![A]\!] \vee [\![B]\!]}\vee \mathrm{R}_2$

1. $\llcorner\Gamma\lrcorner \xrightarrow{k} B$ (i.h. on premise)

2. $\llcorner\Gamma\lrcorner \xrightarrow{k} A \vee B$ (Rule ($\vee$ R$_2$))

Case. $$\frac{\Gamma, [\![A]\!] \vee [\![B]\!], [\![A]\!] \xrightarrow{k} [\![C]\!] \qquad \Gamma, [\![A]\!] \vee [\![B]\!], [\![B]\!] \xrightarrow{k} [\![C]\!]}{\Gamma, [\![A]\!] \vee [\![B]\!] \xrightarrow{k} [\![C]\!]}\vee\mathrm{L}$$

1. $\llcorner\Gamma\lrcorner, A \vee B, A \xrightarrow{k} C$ (i.h. on 1st premise)
2. $\llcorner\Gamma\lrcorner, A \vee B, B \xrightarrow{k} C$ (i.h. on 2nd premise)
3. $\llcorner\Gamma\lrcorner, A \vee B \xrightarrow{k} C$ (Rule ($\vee$L))

Case. $$\frac{}{\Gamma \xrightarrow{k} \top}\top\mathrm{R}$$

1. $\llcorner\Gamma\lrcorner \xrightarrow{k} \top$ (Rule ($\top$R))

Case. $$\frac{}{\Gamma, \bot \xrightarrow{k} [\![C]\!]}\bot\mathrm{L}$$

1. $\llcorner\Gamma\lrcorner, \bot \xrightarrow{k} C$ (Rule ($\bot$L))

Case. $$\frac{\Gamma, [\![A]\!] \xrightarrow{k} [\![B]\!]}{\Gamma \xrightarrow{k} [\![A]\!] \supset [\![B]\!]}\supset\mathrm{R}$$

1. $\llcorner\Gamma\lrcorner, A \xrightarrow{k} B$ (i.h. on premise)
2. $\llcorner\Gamma\lrcorner \xrightarrow{k} A \supset B$ (Rule ($\supset$R))

Case. $$\frac{\Gamma, [\![A]\!] \supset [\![B]\!] \xrightarrow{k} [\![A]\!] \qquad \Gamma, [\![A]\!] \supset [\![B]\!], [\![B]\!] \xrightarrow{k} [\![C]\!]}{\Gamma, [\![A]\!] \supset [\![B]\!] \xrightarrow{k} [\![C]\!]}\supset\mathrm{L}$$

1. $\llcorner\Gamma\lrcorner, A \supset B \xrightarrow{k} A$ (i.h. on 1st premise)
2. $\llcorner\Gamma\lrcorner, A \supset B, B \xrightarrow{k} C$ (i.h. on 2nd premise)
3. $\llcorner\Gamma\lrcorner, A \supset B \xrightarrow{k} C$ (Rule ($\supset$L))

Proof of (2).

Case. $$\frac{\Gamma, \ell \textsf{ claims } k' \textsf{ says } [\![A]\!], k' \textsf{ says } [\![A]\!] \xrightarrow{\ell} k \textsf{ says } [\![C]\!] \qquad \ell \succeq \ell}{\Gamma, \ell \textsf{ claims } k' \textsf{ says } [\![A]\!] \xrightarrow{\ell} k \textsf{ says } [\![C]\!]}\mathrm{claims}$$

1. $\llcorner\Gamma\lrcorner, k' \textsf{ claims } A, k' \textsf{ claims } A \xrightarrow{k} C$ (i.h. on premise)

2. $\llcorner \Gamma \lrcorner, k'$ claims $A \xrightarrow{k} C$ (Strengthening)

Case. $\dfrac{\Gamma|_k \xrightarrow{k} [\![A]\!]}{\Gamma \xrightarrow{\ell} k \text{ says } [\![A]\!]}\text{saysR}$

1. $\llcorner \Gamma|_k \lrcorner \xrightarrow{k} A$ (i.h. (1) on premise)

2. $\llcorner \Gamma \lrcorner \xrightarrow{k} A$ (Weakening)

Case. $\dfrac{\Gamma, k' \text{ says } [\![A]\!], k' \text{ claims } [\![A]\!] \xrightarrow{\ell} k \text{ says } [\![C]\!]}{\Gamma, k' \text{ says } [\![A]\!] \xrightarrow{\ell} k \text{ says } [\![C]\!]}\text{saysL}$

1. $\llcorner \Gamma \lrcorner, k'$ claims A, k' claims $A \xrightarrow{k} C$ (i.h. on premise)

2. $\llcorner \Gamma \lrcorner, k'$ claims $A \xrightarrow{k} C$ (Strengthening)

No other cases apply. □

We now prove Theorems 5.6 and 5.7.

Theorem I.9 (Equivalence; Theorem 5.6)**.** $\cdot \xrightarrow{K} A$ *in BL_0's sequent calculus if and only if* $\vdash K$ says A *in BL_0's axiomatic system.*

Proof. Suppose $\cdot \xrightarrow{K} A$ in BL_0's sequent calculus. By Lemma I.4, $\cdot \vdash_G K$ says $(\top \supset A)$ in BL_0's generalized axiomatic system. By Theorem I.3, $\vdash K$ says $(\top \supset A)$. Now, as in the proof of Corollary C.10, this implies $\vdash K$ says A.

Conversely, suppose that $\vdash K$ says A in BL_0's axiomatic system. By Lemma I.5, $\cdot \xrightarrow{\ell} \ell$ says K says $[\![A]\!]$ in DTL_0. There is only one rule that can applied to derive this: (saysR). Hence, $\cdot \xrightarrow{\ell} K$ says $[\![A]\!]$. Again, only the rule (saysR) can derive this. Thus $\cdot \xrightarrow{K} [\![A]\!]$. Now by Lemma I.8.1, $\cdot \xrightarrow{K} A$ in BL_0's sequent calculus. □

Theorem I.10 (Correctness; Theorem 5.7)**.** $\Gamma \xrightarrow{K} A$ *in BL_0's sequent calculus if and only if* $[\![\Gamma]\!] \xrightarrow{K} [\![A]\!]$ *in DTL_0's sequent calculus.*

Proof. Suppose $\Gamma \xrightarrow{K} A$ in BL_0's sequent calculus. Let $\Gamma = A_1, \ldots, A_n$. Then,

1. $\cdot \vdash_G K$ says $(\overline{\Gamma} \supset A)$ (Lemma I.4)

2. $\vdash K$ says $(\overline{\Gamma} \supset A)$ (Theorem I.3)

3. $\cdot \xrightarrow{\ell} \ell$ says K says $([\![\overline{\Gamma}]\!] \supset [\![A]\!])$ (Lemma I.5)

4. $\cdot \xrightarrow{K} [\![\overline{\Gamma}]\!] \supset [\![A]\!]$ (Inversion)

5. $\cdot \xrightarrow{K} ([\![A_1]\!] \wedge \ldots \wedge [\![A_n]\!]) \supset [\![A]\!]$ (Definitions)

6. $[\![A_1]\!], \ldots, [\![A_n]\!], ([\![A_1]\!] \wedge \ldots \wedge [\![A_n]\!]) \supset [\![A]\!] \xrightarrow{K} [\![A]\!]$ (Basic reasoning)

$$\frac{(A\text{ atomic})}{\Gamma, A \vdash A}\text{init} \qquad \frac{\Box\Gamma \vdash A}{\Box\Gamma, \Gamma' \vdash \Box A}\Box\text{R} \qquad \frac{\Gamma, \Box A, A \vdash C}{\Gamma, \Box A \vdash C}\Box\text{L}$$

$$\frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \wedge B}\wedge\text{R} \qquad \frac{\Gamma, A \wedge B, A \vdash C}{\Gamma, A \wedge B \vdash C}\wedge\text{L}_1 \qquad \frac{\Gamma, A \wedge B, B \vdash C}{\Gamma, A \wedge B \vdash C}\wedge\text{L}_2$$

$$\frac{\Gamma \vdash A}{\Gamma \vdash A \vee B}\vee\text{R}_1 \qquad \frac{\Gamma \vdash B}{\Gamma \vdash A \vee B}\vee\text{R}_2 \qquad \frac{\Gamma, A \vee B, A \vdash C \qquad \Gamma, A \vee B, B \vdash C}{\Gamma, A \vee B \vdash C}\vee\text{L}$$

$$\frac{}{\Gamma \vdash \top}\top\text{R} \qquad \frac{}{\Gamma, \bot \vdash C}\bot\text{L}$$

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash A \supset B}\supset\text{R} \qquad \frac{\Gamma, A \supset B \vdash A \qquad \Gamma, A \supset B, B \vdash C}{\Gamma, A \supset B \vdash C}\supset\text{L}$$

Figure 7: Cut-free Sequent Calculus for CS4 (Taken from [13])

7. $[\![A_1]\!], \ldots, [\![A_n]\!] \xrightarrow{K} [\![A]\!]$ i.e. $[\![\Gamma]\!] \xrightarrow{K} [\![A]\!]$ (Theorem B.2 on 5 and 6)

Conversely, suppose $[\![\Gamma]\!] \xrightarrow{K} [\![A]\!]$. By Lemma I.8.1, $\llcorner[\![\Gamma]\!]\lrcorner \xrightarrow{K} \llcorner[\![A]\!]\lrcorner$. But by definition, $\llcorner[\![\Gamma]\!]\lrcorner = \Gamma$ and $\llcorner[\![A]\!]\lrcorner = A$. Therefore, $\Gamma \xrightarrow{K} A$. □

I.3 Proof of Theorem 5.9

In this section we show that the translation from BL_0 to CS4 is sound and complete (Theorem 5.9). We use a sequent calculus for CS4, shown in Figure 7. Γ denotes a set of formulas in CS4, and $\Box\Gamma$ denotes a set of formulas of the form $\Box A$. This sequent calculus is the $\Diamond$-free fragment of a sequent calculus described by Bierman et al [13], with only the difference that we restrict initial sequents (Rule (init)) to atomic formulas. However, we show (Lemma I.11 below) that this restricted sequent calculus admits the general (init) rule; hence the two formulations of the sequent calculus are equivalent. Our version reduces the technical difficulty of proving completeness of the translation. It is shown in Bierman et al's paper that the sequent calculus is equivalent to the axiomatic formulation described in Section 5.1, that it admits weakening and the cut rule, and that it has the subformula property.

Lemma I.11 (Identity)**.** *For each CS4 formula A, it is the case that $\Gamma, A \vdash A$.*

Proof. By induction on A. We case analyze the top constructor in A. Most cases work as in the proof of Theorem B.3. The case $A = \Box B$ is new, and the case $A = A_1 \wedge A_2$ is different, because in CS4's sequent calculus we use two left rules for $\wedge$, whereas in DTL_0's sequent calculus there is one left rule for $\wedge$. We show these two cases below.

Case. $A = \Box B$. Let $\Gamma = \Box\Gamma', \Gamma''$

1. $\Box\Gamma', \Box B, B \vdash B$ (i.h. on B)

2. $\Box\Gamma', \Box B \vdash B$ (Rule ($\Box$L))

3. $\Box\Gamma', \Gamma'', \Box B \vdash \Box B$ (Rule ($\Box$R))

Case. $A = A_1 \wedge A_2$

1. $\Gamma, A_1 \wedge A_2, A_1 \vdash A_1$ (i.h. on A_1)
2. $\Gamma, A_1 \wedge A_2 \vdash A_1$ (Rule ($\wedge$ L$_1$))
3. $\Gamma, A_1 \wedge A_2, A_2 \vdash A_2$ (i.h. on A_2)
4. $\Gamma, A_1 \wedge A_2 \vdash A_2$ (Rule ($\wedge$ L$_2$))
5. $\Gamma, A_1 \wedge A_2 \vdash A_1 \wedge A_2$ (Rule ($\wedge$ R) on 2 and 4)

$\Box$

Restricting initial sequents to atomic formulas allows us to prove the following inversion theorem, which helps us simplify the completeness proof. We can also prove completeness without this theorem, but we would have to consider many more cases.

Lemma I.12 (Inversion in CS4)**.** *If there is a derivation of* $\Gamma \vdash A \supset B$ *in CS4's sequent calculus, then there is a shorter or equal derivation of* $\Gamma, A \vdash B$*.*

Proof. We induct on the given derivation of $\Gamma \vdash A \supset B$, analyzing cases on the last rule. We do not explicitly prove that the constructed derivations are shorter or equal; the reader may verify this easily in each case.

Case. $\dfrac{\Gamma, \Box C, C \vdash A \supset B}{\Gamma, \Box C \vdash A \supset B}\Box\mathrm{L}$

1. $\Gamma, \Box C, C, A \vdash B$ (i.h. on premise)
2. $\Gamma, \Box C, A \vdash B$ (Rule ($\Box$L))

Case. $\dfrac{\Gamma, C_1 \wedge C_2, C_1 \vdash A \supset B}{\Gamma, C_1 \wedge C_2 \vdash A \supset B}\wedge\mathrm{L}_1$

1. $\Gamma, C_1 \wedge C_2, C_1, A \vdash B$ (i.h. on premise)
2. $\Gamma, C_1 \wedge C_2, A \vdash B$ (Rule ($\wedge$ L$_1$))

Case. $\dfrac{\Gamma, C_1 \wedge C_2, C_2 \vdash A \supset B}{\Gamma, C_1 \wedge C_2 \vdash A \supset B}\wedge\mathrm{L}_2$

1. $\Gamma, C_1 \wedge C_2, C_2, A \vdash B$ (i.h. on premise)
2. $\Gamma, C_1 \wedge C_2, A \vdash B$ (Rule ($\wedge$ L$_2$))

Case. $\dfrac{\Gamma, C_1 \vee C_2, C_1 \vdash A \supset B \qquad \Gamma, C_1 \vee C_2, C_2 \vdash A \supset B}{\Gamma, C_1 \vee C_2 \vdash A \supset B}\vee\mathrm{L}$

1. $\Gamma, C_1 \vee C_2, C_1, A \vdash B$ (i.h. on 1st premise)
2. $\Gamma, C_1 \vee C_2, C_2, A \vdash B$ (i.h. on 2nd premise)
3. $\Gamma, C_1 \vee C_2, A \vdash B$ (Rule ($\vee$L))

Case. $\dfrac{}{\Gamma, \bot \vdash A \supset B}\bot\mathrm{L}$

1. $\Gamma, \bot, A \vdash B$ (Rule ($\bot$L))

Case. $\dfrac{\Gamma, A \vdash B}{\Gamma \vdash A \supset B}\supset\mathrm{R}$

1. $\Gamma, A \vdash B$ (Premise)

Case. $\dfrac{\Gamma, C_1 \supset C_2 \vdash C_1 \qquad \Gamma, C_1 \supset C_2, C_2 \vdash A \supset B}{\Gamma, C_1 \supset C_2 \vdash A \supset B}\supset\mathrm{L}$

1. $\Gamma, C_1 \supset C_2, A \vdash C_1$ (Weakening on 1st premise)
2. $\Gamma, C_1 \supset C_2, C_2, A \vdash B$ (i.h. on 2nd premise)
3. $\Gamma, C_1 \supset C_2, A \vdash B$ (Rule ($\supset$L))

No other cases apply. □

We also need a lemma about BL$_0$'s sequent calculus for proving completeness.

Lemma I.13. *If* $\Gamma, K\ \mathsf{says}\ A \xrightarrow{K'} C$ *in BL$_0$, then there is a shorter or equal derivation of* $\Gamma, K\ \mathsf{claims}\ A \xrightarrow{K'} C$.

Proof. By induction on the derivation of $\Gamma, K\ \mathsf{says}\ A \xrightarrow{K'} C$. □

Lemma I.14 (Soundness of Translation)**.** *If* $\Gamma \xrightarrow{K} A$ *in the sequent calculus of BL$_0$, then* $\ulcorner\Gamma\urcorner, K \vdash \ulcorner A\urcorner$ *in the sequent calculus of CS4 (where the translation of the context* $\ulcorner\Gamma\urcorner$ *is defined pointwise, and* $K\ \mathsf{claims}\ A$ *is treated as* $K\ \mathsf{says}\ A$*).*

Proof. We induct on the derivation of $\Gamma \xrightarrow{K} A$, analyzing cases of the last rule.

Case. $\dfrac{P \text{ atomic}}{\Gamma, P \xrightarrow{K} P}\mathrm{init}$

1. $\ulcorner\Gamma\urcorner, P, K \vdash P$ (Rule (init))

Case. $\dfrac{\Gamma, K\ \mathsf{claims}\ A, A \xrightarrow{K} C}{\Gamma, K\ \mathsf{claims}\ A \xrightarrow{K} C}\mathrm{claims}$

1. $\ulcorner\Gamma\urcorner, \Box(K \supset \ulcorner A\urcorner), K \supset \ulcorner A\urcorner, K \vdash K$ (Rule (init))

2. $\ulcorner\Gamma\urcorner, \Box(K \supset \ulcorner A\urcorner), K \supset \ulcorner A\urcorner, K, \ulcorner A\urcorner \vdash \ulcorner A\urcorner$ (Lemma I.11)

3. $\ulcorner\Gamma\urcorner, \Box(K \supset \ulcorner A\urcorner), K \supset \ulcorner A\urcorner, K \vdash \ulcorner A\urcorner$ (Rule ($\supset$L))

4. $\ulcorner\Gamma\urcorner, \Box(K \supset \ulcorner A\urcorner), K \vdash \ulcorner A\urcorner$ (Rule ($\Box$L))

5. $\ulcorner\Gamma\urcorner, \Box(K \supset \ulcorner A\urcorner), \ulcorner A\urcorner, K \vdash \ulcorner C\urcorner$ (i.h. on premise)

6. $\ulcorner\Gamma\urcorner, \Box(K \supset \ulcorner A\urcorner), K \vdash \ulcorner C\urcorner$ (Cut on 4 and 5)

Case. $\dfrac{\Gamma| \xrightarrow{K} A}{\Gamma \xrightarrow{K'} K \textsf{ says } A}\textsf{says}\mathrm{R}$

Let $\Gamma| = K_1 \textsf{ claims } A_1, \ldots, K_n \textsf{ claims } A_n$.

1. $\Box(K_1 \supset \ulcorner A_1\urcorner), \ldots, \Box(K_n \supset \ulcorner A_n\urcorner), K \vdash \ulcorner A\urcorner$ (i.h. on premise)

2. $\Box(K_1 \supset \ulcorner A_1\urcorner), \ldots, \Box(K_n \supset \ulcorner A_n\urcorner) \vdash K \supset \ulcorner A\urcorner$ (Rule ($\supset$R))

3. $\Box(K_1 \supset \ulcorner A_1\urcorner), \ldots, \Box(K_n \supset \ulcorner A_n\urcorner) \vdash \Box(K \supset \ulcorner A\urcorner)$ (Rule ($\Box$R))

4. $\ulcorner\Gamma\urcorner, K' \vdash \Box(K \supset \ulcorner A\urcorner)$ (Weakening)

Case. $\dfrac{\Gamma, K \textsf{ says } A, K \textsf{ claims } A \xrightarrow{K'} C}{\Gamma, K \textsf{ says } A \xrightarrow{K'} C}\textsf{says}\mathrm{L}$

1. $\ulcorner\Gamma\urcorner, \Box(K \supset \ulcorner A\urcorner), \Box(K \supset \ulcorner A\urcorner), K' \vdash \ulcorner C\urcorner$ (i.h. on premise)

2. $\ulcorner\Gamma\urcorner, \Box(K \supset \ulcorner A\urcorner), K' \vdash \ulcorner C\urcorner$ (Strengthening)

Case. $\dfrac{\Gamma \xrightarrow{K} A \qquad \Gamma \xrightarrow{K} B}{\Gamma \xrightarrow{K} A \wedge B}\wedge\mathrm{R}$

1. $\ulcorner\Gamma\urcorner, K \vdash \ulcorner A\urcorner$ (i.h. on 1st premise)

2. $\ulcorner\Gamma\urcorner, K \vdash \ulcorner B\urcorner$ (i.h. on 2nd premise)

3. $\ulcorner\Gamma\urcorner, K \vdash \ulcorner A\urcorner \wedge \ulcorner B\urcorner$ (Rule ($\wedge$R))

Case. $\dfrac{\Gamma, A \wedge B, A, B \xrightarrow{K} C}{\Gamma, A \wedge B \xrightarrow{K} C}\wedge\mathrm{L}$

1. $\ulcorner\Gamma\urcorner, \ulcorner A\urcorner \wedge \ulcorner B\urcorner, \ulcorner A\urcorner, \ulcorner B\urcorner, K \vdash \ulcorner C\urcorner$ (i.h. on premise)

2. $\ulcorner\Gamma\urcorner, \ulcorner A\urcorner \wedge \ulcorner B\urcorner, \ulcorner A\urcorner, \ulcorner B\urcorner, K \vdash \ulcorner A\urcorner$ (Lemma I.11)

3. $\ulcorner\Gamma\urcorner, \ulcorner A\urcorner \wedge \ulcorner B\urcorner, \ulcorner B\urcorner, K \vdash \ulcorner A\urcorner$ (Rule ($\wedge$ L_1))

4. $\ulcorner\Gamma\urcorner, \ulcorner A\urcorner \wedge \ulcorner B\urcorner, \ulcorner B\urcorner, K \vdash \ulcorner B\urcorner$ (Lemma I.11)

5. $\ulcorner\Gamma\urcorner, \ulcorner A\urcorner \wedge \ulcorner B\urcorner, K \vdash \ulcorner B\urcorner$ (Rule ($\wedge$ L_2))

6. $\ulcorner\Gamma\urcorner, \ulcorner A\urcorner \wedge \ulcorner B\urcorner, \ulcorner B\urcorner, K \vdash \ulcorner C\urcorner$ (Cut on 3 and 1)

7. $\ulcorner\Gamma\urcorner, \ulcorner A\urcorner \wedge \ulcorner B\urcorner, K \vdash \ulcorner C\urcorner$ (Cut on 5 and 6)

Case. $\dfrac{\Gamma \xrightarrow{K} A}{\Gamma \xrightarrow{K} A \vee B} \vee \mathrm{R}_1$

1. $\ulcorner\Gamma\urcorner, K \vdash \ulcorner A\urcorner$ (i.h. on premise)
2. $\ulcorner\Gamma\urcorner, K \vdash \ulcorner A\urcorner \vee \ulcorner B\urcorner$ (Rule ($\vee$ R_1))

Case. $\dfrac{\Gamma \xrightarrow{K} B}{\Gamma \xrightarrow{K} A \vee B} \vee \mathrm{R}_2$

1. $\ulcorner\Gamma\urcorner, K \vdash \ulcorner B\urcorner$ (i.h. on premise)
2. $\ulcorner\Gamma\urcorner, K \vdash \ulcorner A\urcorner \vee \ulcorner B\urcorner$ (Rule ($\vee$ R_2))

Case. $\dfrac{\Gamma, A \vee B, A \xrightarrow{K} C \qquad \Gamma, A \vee B, B \xrightarrow{K} C}{\Gamma, A \vee B \xrightarrow{K} C} \vee \mathrm{L}$

1. $\ulcorner\Gamma\urcorner, \ulcorner A\urcorner \vee \ulcorner B\urcorner, \ulcorner A\urcorner, K \vdash \ulcorner C\urcorner$ (i.h. on 1st premise)
2. $\ulcorner\Gamma\urcorner, \ulcorner A\urcorner \vee \ulcorner B\urcorner, \ulcorner B\urcorner, K \vdash \ulcorner C\urcorner$ (i.h. on 2nd premise)
3. $\ulcorner\Gamma\urcorner, \ulcorner A\urcorner \vee \ulcorner B\urcorner, K \vdash \ulcorner C\urcorner$ (Rule ($\vee$L))

Case. $\dfrac{}{\Gamma \xrightarrow{K} \top} \top\mathrm{R}$

1. $\ulcorner\Gamma\urcorner, K \vdash \top$ (Rule ($\top$R))

Case. $\dfrac{}{\Gamma, \bot \xrightarrow{K} C} \bot\mathrm{L}$

1. $\ulcorner\Gamma\urcorner, \bot, K \vdash \ulcorner C\urcorner$ (Rule ($\bot$L))

Case. $\dfrac{\Gamma, A \xrightarrow{K} B}{\Gamma \xrightarrow{K} A \supset B} \supset\mathrm{R}$

1. $\ulcorner\Gamma\urcorner, \ulcorner A\urcorner, K \vdash \ulcorner B\urcorner$ (i.h. on premise)
2. $\ulcorner\Gamma\urcorner, K \vdash \ulcorner A\urcorner \supset \ulcorner B\urcorner$ (Rule ($\supset$R))

Case. $\dfrac{\Gamma, A \supset B \xrightarrow{K} A \qquad \Gamma, A \supset B, B \xrightarrow{K} C}{\Gamma, A \supset B \xrightarrow{K} C} \supset\mathrm{L}$

1. $\ulcorner\Gamma\urcorner, \ulcorner A\urcorner \supset \ulcorner B\urcorner, K \vdash \ulcorner A\urcorner$ (i.h. on 1st premise)

2. $\ulcorner\Gamma\urcorner, \ulcorner A\urcorner \supset \ulcorner B\urcorner, \ulcorner B\urcorner, K \vdash \ulcorner C\urcorner$ (i.h. on 2nd premise)

3. $\ulcorner\Gamma\urcorner, \ulcorner A\urcorner \supset \ulcorner B\urcorner, K \vdash \ulcorner C\urcorner$ (Rule ($\supset$L))

$\square$

Next, we seek to show the converse of the above lemma, namely, if $\ulcorner\Gamma\urcorner, K \vdash \ulcorner A\urcorner$ in CS4, then $\Gamma \xrightarrow{K} A$ in BL_0. Our approach is based on a careful characterization of sequents that may occur in a proof of $\ulcorner\Gamma\urcorner, K \vdash \ulcorner A\urcorner$. We call such sequents regular sequents.

Definition I.15 (Regular Hypothesis)**.** A CS4 hypothesis Γ is called regular if it contains formulas of the form $\ulcorner A\urcorner$ and $K \supset \ulcorner A\urcorner$ only (A denotes an arbitrary BL_0 formula).

Definition I.16 (Regular Sequents)**.** A CS4 sequent is called regular if it has the form $\Gamma, K \vdash \ulcorner A\urcorner$, where Γ is a regular hypothesis.

Next, we define an inverse translation from regular sequents to BL_0 sequents.

Definition I.17 (Inverse Translation)**.** The inverse translation $\llcorner\cdot\lrcorner$ for regular hypothesis is defined pointwise, where the inverse translation for formulas is defined as follows.

$$\begin{array}{rcl} \llcorner\ulcorner A\urcorner\lrcorner & = & A \\ \llcorner K \supset \ulcorner A\urcorner\lrcorner & = & K \textsf{ claims } A \end{array}$$

A regular sequent $\Gamma, K \vdash \ulcorner A\urcorner$ is inverse translated to $\llcorner\Gamma\lrcorner \xrightarrow{K} A$.

The following completeness lemma contains two statements that we prove by simultaneous induction. The second statement is the actual completeness that we need. The first statement is needed for induction to work.

Lemma I.18 (Completeness of Translation)**.** *Let Γ be a regular hypothesis.*

1. *If $\Gamma, K \vdash K'$ in CS4 and $K \neq K'$, then $\llcorner\Gamma\lrcorner \xrightarrow{K} C$ for any C in BL_0.*

2. *If $\Gamma, K \vdash \ulcorner A\urcorner$ in CS4, then $\llcorner\Gamma\lrcorner \xrightarrow{K} A$ in BL_0.*

Proof. We induct on the *depth* of the given derivations, and analyze cases of the last rule in the derivations.

Proof of (1).

Case. $\dfrac{}{\Gamma, K \vdash K'}\text{init}$

1. $K = K'$ ($K' \notin \Gamma$ by regularity)

2. Contradiction (Assumption $K \neq K'$ and 1)

3. $\llcorner\Gamma\lrcorner \xrightarrow{K} C$ (RAA)

Case. $\dfrac{\Gamma, \Box(K'' \supset \ulcorner A \urcorner), K'' \supset \ulcorner A \urcorner, K \vdash K'}{\Gamma, \Box(K'' \supset \ulcorner A \urcorner), K \vdash K'}\Box\mathrm{L}$

1. $\llcorner \Gamma \lrcorner, K''$ says A, K'' claims $A \xrightarrow{K} C$ (i.h. on premise)
2. $\llcorner \Gamma \lrcorner, K''$ says $A \xrightarrow{K} C$ (Rule (saysL))

Case. $\dfrac{\Gamma, \ulcorner A \urcorner \wedge \ulcorner B \urcorner, \ulcorner A \urcorner, K \vdash K'}{\Gamma, \ulcorner A \urcorner \wedge \ulcorner B \urcorner, K \vdash K'}\wedge \mathrm{L}_1$

1. $\llcorner \Gamma \lrcorner, A \wedge B, A \xrightarrow{K} C$ (i.h. on premise)
2. $\llcorner \Gamma \lrcorner, A \wedge B, A, B \xrightarrow{K} C$ (Weakening)
3. $\llcorner \Gamma \lrcorner, A \wedge B \xrightarrow{K} C$ (Rule ($\wedge$L))

Case. $\dfrac{\Gamma, \ulcorner A \urcorner \wedge \ulcorner B \urcorner, \ulcorner B \urcorner, K \vdash K'}{\Gamma, \ulcorner A \urcorner \wedge \ulcorner B \urcorner, K \vdash K'}\wedge \mathrm{L}_2$

1. $\llcorner \Gamma \lrcorner, A \wedge B, B \xrightarrow{K} C$ (i.h. on premise)
2. $\llcorner \Gamma \lrcorner, A \wedge B, A, B \xrightarrow{K} C$ (Weakening)
3. $\llcorner \Gamma \lrcorner, A \wedge B \xrightarrow{K} C$ (Rule ($\wedge$L))

Case. $\dfrac{\Gamma, \ulcorner A \urcorner \vee \ulcorner B \urcorner, \ulcorner A \urcorner, K \vdash K' \qquad \Gamma, \ulcorner A \urcorner \vee \ulcorner B \urcorner, \ulcorner B \urcorner, K \vdash K'}{\Gamma, \ulcorner A \urcorner \vee \ulcorner B \urcorner, K \vdash K'}\vee\mathrm{L}$

1. $\llcorner \Gamma \lrcorner, A \vee B, A \xrightarrow{K} C$ (i.h. on 1st premise)
2. $\llcorner \Gamma \lrcorner, A \vee B, B \xrightarrow{K} C$ (i.h. on 2nd premise)
3. $\llcorner \Gamma \lrcorner, A \vee B \xrightarrow{K} C$ (Rule ($\vee$L))

Case. $\dfrac{}{\Gamma, \bot, K \vdash K'}\bot\mathrm{L}$

1. $\llcorner \Gamma \lrcorner, \bot \xrightarrow{K} C$ (Rule ($\bot$L))

Case. $\dfrac{\Gamma, \ulcorner A \urcorner \supset \ulcorner B \urcorner, K \vdash \ulcorner A \urcorner \qquad \Gamma, \ulcorner A \urcorner \supset \ulcorner B \urcorner, \ulcorner B \urcorner, K \vdash K'}{\Gamma, \ulcorner A \urcorner \supset \ulcorner B \urcorner, K \vdash K'}\supset\mathrm{L}$

1. $\llcorner \Gamma \lrcorner, A \supset B \xrightarrow{K} A$ (i.h. (2) on 1st premise)
2. $\llcorner \Gamma \lrcorner, A \supset B, B \xrightarrow{K} C$ (i.h. on 2nd premise)
3. $\llcorner \Gamma \lrcorner, A \supset B \xrightarrow{K} C$ (Rule ($\supset$L))

Case. $$\frac{\Gamma, K'' \supset \ulcorner B \urcorner, K \vdash K'' \qquad \Gamma, K'' \supset \ulcorner B \urcorner, \ulcorner B \urcorner, K \vdash K'}{\Gamma, K'' \supset \ulcorner B \urcorner, K \vdash K'}\supset\mathrm{L}$$

We analyze two subcases:

Case. $K = K''$

1. $\llcorner \Gamma \lrcorner, K''$ claims $B, B \xrightarrow{K} C$ (i.h. on the 2nd premise)
2. $\llcorner \Gamma \lrcorner, K''$ claims $B \xrightarrow{K} C$ (Rule (claims); $K = K''$)

Case. $K \neq K''$

1. $\llcorner \Gamma \lrcorner, K''$ claims $B \xrightarrow{K} C$ (i.h. on 1st premise)

Proof of (2).

Case. $$\frac{(A \text{ atomic})}{\Gamma, A, K \vdash A}\text{init}$$

1. $\llcorner \Gamma \lrcorner, A \xrightarrow{K} A$ (Rule (init))

Case. $$\frac{\Box\Gamma \vdash K' \supset \ulcorner A \urcorner}{\Box\Gamma, \Gamma', K \vdash \Box(K' \supset \ulcorner A \urcorner)}\Box\mathrm{R}$$

Let $\Gamma = \Box(K_1 \supset \ulcorner A_1 \urcorner), \ldots, \Box(K_n \supset \ulcorner A_n \urcorner)$.

1. $\Box\Gamma, K' \vdash \ulcorner A \urcorner$ (Lemma I.12 on premise)
2. K_1 says $A_1, \ldots, K_n$ says $A_n \xrightarrow{K'} A$ (i.h. on 1)
3. K_1 claims $A_1, \ldots, K_n$ claims $A_n \xrightarrow{K'} A$ (Lemma I.13)
4. K_1 claims $A_1, \ldots, K_n$ claims $A_n, \llcorner \Gamma' \lrcorner \xrightarrow{K} K'$ says A (Rule (saysR))
5. K_1 says $A_1, \ldots, K_n$ says $A_n, \llcorner \Gamma' \lrcorner \xrightarrow{K} K'$ says A (Rule (saysL))

Case. $$\frac{\Gamma, \Box(K' \supset \ulcorner A \urcorner), K' \supset \ulcorner A \urcorner, K \vdash \ulcorner C \urcorner}{\Gamma, \Box(K' \supset \ulcorner A \urcorner), K \vdash \ulcorner C \urcorner}\Box\mathrm{L}$$

1. $\llcorner \Gamma \lrcorner, K'$ says A, K' claims $A \xrightarrow{K} C$ (i.h. on premise)
2. $\llcorner \Gamma \lrcorner, K'$ says $A \xrightarrow{K} C$ (Rule (saysL))

Case. $$\frac{\Gamma, K \vdash \ulcorner A \urcorner \qquad \Gamma, K \vdash \ulcorner B \urcorner}{\Gamma, K \vdash \ulcorner A \urcorner \wedge \ulcorner B \urcorner}\wedge\mathrm{R}$$

1. $\llcorner \Gamma \lrcorner \xrightarrow{K} A$ (i.h. on 1st premise)
2. $\llcorner \Gamma \lrcorner \xrightarrow{K} B$ (i.h. on 2nd premise)

3. $\llcorner\Gamma\lrcorner \xrightarrow{K} A \wedge B$ (Rule ($\wedge$R))

Case. $\dfrac{\Gamma, \ulcorner A\urcorner \wedge \ulcorner B\urcorner, \ulcorner A\urcorner, K \vdash \ulcorner C\urcorner}{\Gamma, \ulcorner A\urcorner \wedge \ulcorner B\urcorner, K \vdash \ulcorner C\urcorner} \wedge \mathrm{L}_1$

1. $\llcorner\Gamma\lrcorner, A \wedge B, A \xrightarrow{K} C$ (i.h. on premise)
2. $\llcorner\Gamma\lrcorner, A \wedge B, A, B \xrightarrow{K} C$ (Weakening)
3. $\llcorner\Gamma\lrcorner, A \wedge B \xrightarrow{K} C$ (Rule ($\wedge$L))

Case. $\dfrac{\Gamma, \ulcorner A\urcorner \wedge \ulcorner B\urcorner, \ulcorner B\urcorner, K \vdash \ulcorner C\urcorner}{\Gamma, \ulcorner A\urcorner \wedge \ulcorner B\urcorner, K \vdash \ulcorner C\urcorner} \wedge \mathrm{L}_2$

1. $\llcorner\Gamma\lrcorner, A \wedge B, B \xrightarrow{K} C$ (i.h. on premise)
2. $\llcorner\Gamma\lrcorner, A \wedge B, A, B \xrightarrow{K} C$ (Weakening)
3. $\llcorner\Gamma\lrcorner, A \wedge B \xrightarrow{K} C$ (Rule ($\wedge$L))

Case. $\dfrac{\Gamma, K \vdash \ulcorner A\urcorner}{\Gamma, K \vdash \ulcorner A\urcorner \vee \ulcorner B\urcorner} \vee \mathrm{R}_1$

1. $\llcorner\Gamma\lrcorner \xrightarrow{K} A$ (i.h. on premise)
2. $\llcorner\Gamma\lrcorner \xrightarrow{K} A \vee B$ (Rule ($\vee$ R_1))

Case. $\dfrac{\Gamma, K \vdash \ulcorner B\urcorner}{\Gamma, K \vdash \ulcorner A\urcorner \vee \ulcorner B\urcorner} \vee \mathrm{R}_2$

1. $\llcorner\Gamma\lrcorner \xrightarrow{K} B$ (i.h. on premise)
2. $\llcorner\Gamma\lrcorner \xrightarrow{K} A \vee B$ (Rule ($\vee$ R_2))

Case. $\dfrac{\Gamma, \ulcorner A\urcorner \vee \ulcorner B\urcorner, \ulcorner A\urcorner, K \vdash \ulcorner C\urcorner \qquad \Gamma, \ulcorner A\urcorner \vee \ulcorner B\urcorner, \ulcorner B\urcorner, K \vdash \ulcorner C\urcorner}{\Gamma, \ulcorner A\urcorner \vee \ulcorner B\urcorner, K \vdash \ulcorner C\urcorner} \vee \mathrm{L}$

1. $\llcorner\Gamma\lrcorner, A \vee B, A \xrightarrow{K} C$ (i.h. on 1st premise)
2. $\llcorner\Gamma\lrcorner, A \vee B, B \xrightarrow{K} C$ (i.h. on 2nd premise)
3. $\llcorner\Gamma\lrcorner, A \vee B \xrightarrow{K} C$ (Rule ($\vee$L))

Case. $\dfrac{}{\Gamma, K \vdash \top} \top\mathrm{R}$

1. $\llcorner\Gamma\lrcorner \xrightarrow{K} \top$ (Rule ($\top$R))

Case. $$\frac{}{\Gamma, \bot, K \vdash \ulcorner C \urcorner}\bot\mathrm{L}$$

1. $\llcorner\Gamma\lrcorner, \bot \xrightarrow{K} C$ (Rule ($\bot$L))

Case. $$\frac{\Gamma, \ulcorner A \urcorner, K \vdash \ulcorner B \urcorner}{\Gamma, K \vdash \ulcorner A \urcorner \supset \ulcorner B \urcorner}\supset\mathrm{R}$$

1. $\llcorner\Gamma\lrcorner, A \xrightarrow{K} B$ (i.h. on premise)
2. $\llcorner\Gamma\lrcorner \xrightarrow{K} A \supset B$ (Rule ($\supset$R))

Case. $$\frac{\Gamma, \ulcorner A \urcorner \supset \ulcorner B \urcorner, K \vdash \ulcorner A \urcorner \qquad \Gamma, \ulcorner A \urcorner \supset \ulcorner B \urcorner, \ulcorner B \urcorner, K \vdash \ulcorner C \urcorner}{\Gamma, \ulcorner A \urcorner \supset \ulcorner B \urcorner, K \vdash \ulcorner C \urcorner}\supset\mathrm{L}$$

1. $\llcorner\Gamma\lrcorner, A \supset B \xrightarrow{K} A$ (i.h. on 1st premise)
2. $\llcorner\Gamma\lrcorner, A \supset B, B \xrightarrow{K} C$ (i.h. on 2nd premise)
3. $\llcorner\Gamma\lrcorner, A \supset B \xrightarrow{K} C$ (Rule ($\supset$L))

Case. $$\frac{\Gamma, K' \supset \ulcorner A \urcorner, K \vdash K' \qquad \Gamma, K' \supset \ulcorner A \urcorner, \ulcorner A \urcorner, K \vdash \ulcorner C \urcorner}{\Gamma, K' \supset \ulcorner A \urcorner, K \vdash \ulcorner C \urcorner}\supset\mathrm{L}$$

We analyze two subcases:

Case. $K = K'$

1. $\llcorner\Gamma\lrcorner, K'\ \mathsf{claims}\ A, A \xrightarrow{K} C$ (i.h. on 2nd premise)
2. $\llcorner\Gamma\lrcorner, K'\ \mathsf{claims}\ A \xrightarrow{K} C$ (Rule (claims); $K = K'$)

Case. $K \neq K'$

1. $\llcorner\Gamma\lrcorner, K'\ \mathsf{claims}\ A \xrightarrow{K} C$ (i.h. (1) on 1st premise)

□

Theorem I.19 (Correctness; Theorem 5.9)**.** *$\cdot \xrightarrow{K} A$ in BL_0 if and only if $\vdash K \supset \ulcorner A \urcorner$ in CS4.*

Proof. Suppose $\cdot \xrightarrow{K} A$ in BL_0. By Lemma I.14, $K \vdash \ulcorner A \urcorner$. Hence by rule ($\supset$R), $\cdot \vdash K \supset \ulcorner A \urcorner$.

Conversely, suppose $\cdot \vdash K \supset \ulcorner A \urcorner$ in CS4. By Lemma I.12, $K \vdash \ulcorner A \urcorner$. By Lemma I.18.2, $\cdot \xrightarrow{K} A$. □